ISRAEL JOURNAL OF MATHEMATICS, Vol. 43, No. 3, 1982

S-FORCING, I.
A "BLACK-BOX" THEOREM FOR MORASSES, WITH APPLICATIONS TO SUPER-SOUSLIN TREES

BY
S. SHELAH[†] AND L. STANLEY

ABSTRACT

We formulate, for regular $\mu > \omega$, a "forcing principle" S_μ which we show is equivalent to the existence of morasses, thus providing a new and systematic method for obtaining applications of morasses. Various examples are given, notably that for infinite κ, if $2^\kappa = \kappa^+$ and there exists a $(\kappa^+, 1)$-morass, then there exists a κ^{++}-super-Souslin tree: a normal κ^{++} tree characterized by a highly absolute "positive" property, and which has a κ^{++}-Souslin subtree. As a consequence we show that $\mathrm{CH} + \mathrm{SH}_{\aleph_2} \Rightarrow \aleph_2$ is $(\text{inaccessible})^L$.

§1. Introduction

The first proofs of the existence of Souslin trees in L used forcing techniques. Then the formulation of $\diamond$ provided an alternative point of view, recapitulating in abstract combinatorial form the essence of the condensation arguments which make the forcing techniques work. This point of view quickly became predominant and was pushed further and further by Jensen, arriving finally at the notion of morass, which, in a very real sense, is a repository of much of the combinatorial structure of L. In the meantime, there developed a widespread conviction, supported by ever more empirical evidence, and nicely summed up in popular heuristic observations of the sort:

(*): "any combinatorial property which is provably consistent by forcing is already true in L (and conversely)".

In this paper we formulate and prove a mathematically precise (and thus necessarily restrictive) version of this remark which reunites the "forcing point of view" and the "combinatorial principle point of view" which had seemed to

[†] This author thanks the US–Israel Binational Science Foundation for partial support of this research.

Received July 9, 1980 and in revised form December 24, 1981

part company in the post-$\diamond$ era (for recent work in a similar vein, see [10], and, in a rather different vein, [1], corollary 5, p. 101).

More precisely, for each regular $\mu > \omega$, we define, in §3, a class of partial orders $\mathcal{S}_\mu$. For $\mathbf{P} \in \mathcal{S}_\mu$, we introduce, also in §3, the notions of uniform dense subsets $D \subseteq \mathbf{P}$ and ideals G which meet D uniformly, in terms of the defining properties of $\mathcal{S}_\mu$. These notions seek to formalize the intuitive notion of a "sufficiently-generic set", and the examples we give of our method show that it's a reasonable one. In the sequel to this paper we deal with strengthenings of this notion which correspond to stronger combinatorial properties. Shelah introduced the forcing principle:

S_μ: If $\mathbf{P} \in \mathcal{S}_\mu$ and $\mathcal{D}$ is a collection of $\leqq \mu$ uniform dense subsets of $\mathbf{P}$, there is an ideal G meeting all members of $\mathcal{D}$ uniformly.

In §§5–6 we prove our main theorem:

THEOREM 1. *For all regular $\mu > \omega$, S_μ iff there is a $(\mu, 1)$-morass.*

Theorem 1 provides a new and systematic method for obtaining applications of morasses: first find a $\mathbf{P} \in \mathcal{S}_\mu$ which does the job, and then check that it suffices to have an ideal G which uniformly meets $\leqq \mu$ uniform dense sets which provably exist. Aside from the intrinsic interest of Theorem 1 and the applications we obtain, our hope is that Theorem 1 will prove to be a useful tool for the working set-theorist/model-theorist/combinatorist/general topologist whose daily fare includes forcing but who has hesitated to master morasses.

Before introducing $\mathcal{S}_\mu$, uniform dense sets and filters meeting them uniformly in the abstract setting, in §2, we give a motivating example, which will also provide the principal application of the method of Theorem 1. We present, for infinite cardinals κ, κ^{++}-super-Souslin trees and the conditions for adding these by forcing (both notions are due to Shelah). The bulk of §2 is devoted to proving for $\mathbf{P}$ the properties which, in §3, will figure in the definition of $\mathcal{S}_{\kappa^+}$, thus showing, by one direction of Theorem 1:

(1) If $2^\kappa = \kappa^+$ and there's a $(\kappa^+, 1)$-morass there's a κ^{++}-super-Souslin tree.

Since a κ^{++}-super-Souslin tree has a κ^{++}-Souslin subtree (see (2.3)), (1) permits us to prove, in (2.16):

THEOREM 2. $\mathrm{CH} + \mathrm{SH}_{\aleph_2} \Rightarrow \aleph_2$ *is (inaccessible)$^{\mathrm{L}}$.*

Theorem 2 should be compared with Corollary 4 and (4), below, and with an earlier result of Gregory [5]:

(2) $$\mathrm{CH} + 2^{\aleph_1} = \aleph_2 + \mathrm{SH}_{\aleph_2} \Rightarrow \aleph_2 \text{ is } (\mathrm{Mahlo})^L.$$

It is not known whether the hypotheses of (2) are consistent relative to, say, reasonable large cardinal hypotheses.

§4 is devoted to further examples: the countable conditions for forcing $\square_{\aleph_1}$, and a modification, due to Shelah, of Burgess's 2-gap–2-cardinal conditions of [2]. The former will also figure in §7. In §5, we prove the left-to-right implication of Theorem 1. We exhibit a **P** and a collection of $\leqq \mu$ uniform dense subsets such that a filter meeting them uniformly guarantees the existence of a $(\mu, 1)$-morass. The definition of **P** is due to D. Velleman. See the historical remarks (1.2) for more on this and the other connections between Velleman's work and ours. The right-to-left implication of Theorem 1 is proved in §6. The construction of the sufficiently generic set is inspired by Jensen's original construction (which it now subsumes, by §4) for regular μ, from enough GCH and a $(\mu^+, 1)$-morass, of a (μ^{++}, μ)-model of a T having a (λ^{++}, λ)-model for some infinite cardinal λ.

(1.1) *Preliminaries*

This paper is divided into seven sections, each of which is divided into several subsections. The nth subsection of the mth section is referred to as $(m.n)$.

Our set theory is ZFC. CH is the continuum hypothesis. As usual, cardinals are initial ordinals, but when we wish to emphasize the "cardinal character" we use $\aleph$'s, while when we wish to emphasize the "ordinal character" we use ω's. μ is always a regular uncountable cardinal, κ is always a cardinal, usually infinite, λ is always a limit ordinal, sometimes a cardinal, α, β, γ, δ, η, ν, ξ, ζ and various decorations of these are always ordinals. SH_κ is the κ-Souslin hypothesis: there are no κ-Souslin trees.

The bulk of our notation and terminology is intended to be standard, or have a clear meaning (e.g. card X for the cardinality of X, o.t. X for the order-type of X, cf α for the cofinality of α). For cardinals λ, $\kappa^{<\lambda}$ is the weak power of κ by $\lambda = \Sigma_{\kappa'<\lambda}\, \kappa^{\kappa'}$.

The following list will hopefully cover all non-standard notation as well as our abuses of notation and language, and the places where we require certain notation to serve double-duty.

A closed unbounded set is a club set. $[X]^\kappa$ is the set of subsets of X having cardinality κ. $[X]^{<\kappa}$, $[X]^{\leq\kappa}$ have the obvious meanings. If α is an ordinal, and if X is a set of ordinals, or if X has a natural well-ordering which is clear from context, we also use $[X]^\alpha$, to refer to the set of subsets of X which have order-type equal to α. Similarly for $[X]^{<\alpha}$, $[X]^{\leqq\alpha}$. The intended meaning should be clear from the context. When we mean order-type, we shall happily confuse a set with its increasing enumeration, when it serves our purposes to do so. We use the usual interval notation for ordinals: thus e.g. $[\alpha,\beta)=\{\gamma:\alpha\leqq\gamma<\beta\}$.

α^n denotes ordinal exponentiation and not n-fold Cartesian product. For sets of ordinals a, b, a^* denotes the set of limit points of a, and $a \ll b$ means that $\inf b$ is greater than all members of a. $\{a,b\}$ has the strong Δ-property just in case there are e, a', b' such that $a=e\cup a'$, $b=e\cup b'$, $e\ll a'$, $e\ll b'$, and either $a'\ll b'$ or $b'\ll a'$. Assuming, say, that $a'\ll b'$ and that $a\neq b$, and letting s, s' be the increasing enumerations of a, b respectively, there is $\gamma\in\operatorname{dom} s\cap\operatorname{dom} s'$ such that $s\restriction\gamma=s'\restriction\gamma$, but $s_i<s'_\gamma$, for all $i\in\operatorname{dom} s$. This γ is denoted $\gamma(s,s')$ and is such that $s_\gamma=\inf a'$, $s'_\gamma=\inf b'$. A set X of sets of ordinals is a strong Δ-system if $a,b\in X\Rightarrow\{a,b\}$ has the strong Δ-property. We heavily use the following standard result:

(*) If μ is regular, if $2^{<\mu}=\mu$, if $X\subseteq[\mu^+]^{<\mu}$ and $\operatorname{card} X\geqq\mu^+$ there is $X'\in[X]^{\mu^+}$ which is a strong Δ-system.

For sequences, s, $\lg s$ is the length of s, i.e. the order-type of $\operatorname{dom} s$ (since for us a sequence is a function whose domain has a natural well-ordering and usually is a set of ordinals). We also use lg in a conventional, but related, way. Lim is the class of limit ordinals, which we take not to contain 0, but in §§3, 6 Lim has another conventional meaning. No confusion should arise. By an increasing function we mean what is sometimes called monotone, or order-preserving, i.e. an ordinal-valued function f on a set of ordinals such that for $\alpha,\beta\in\operatorname{dom} f$, $\alpha<\beta\Rightarrow f(\alpha)<f(\beta)$; for non-decreasing, replace $<$ by $\leqq$. A regressive function on the other hand is, as usual, a function f such that for $\alpha\in\operatorname{dom} f$, $f(\alpha)<\alpha$. α, β are the *same kind* of ordinal iff $\alpha\in\mathrm{Lim}\Leftrightarrow\beta\in\mathrm{Lim}$ and $\alpha=0$ iff $\beta=0$. An ordinal-valued function f on a set of ordinals is *nice* just in case f is increasing, $0\in\operatorname{dom} f$ and $f(0)=0$, for $\lambda\in\operatorname{dom} f$, $f(\lambda)\in\mathrm{Lim}$, and if α, $\alpha+1\in\operatorname{dom} f$, then $f(\alpha+1)=f(\alpha)+1$. In most of the paper, S_μ refers to the principle introduced in §3. However, in (1.3) and §6, S_μ refers to a morass notion defined in (1.3). No confusion should result.

When forcing, $p\geqq q$ means p gives *more* information than q; to emphasize this we write $\mathbf{P}=(P,\geqq)$ which is how we denote practically any and all partial orderings. Also, κ-closed means "$<\kappa$-closed": i.e. any increasing sequence of length $<\kappa$ has an upper bound. $D\subseteq P$ is directed if $p,q\in D\Rightarrow(\exists r\in D)$ $(p,q\leqq r)$. $\mathbf{P}$ is κ-directed-closed if any directed $D\subseteq P$ with $\operatorname{card} D<\kappa$ has an upper bound. It is well-known and easy to prove:

(**) **P** is κ-closed with least upper bounds $\Rightarrow$ **P** is κ-directed-closed $\Rightarrow$ P is κ-closed. Further, if **P** is $\aleph_1$-closed, then **P** is $\aleph_1$-directed-closed.

(1.2) *Historical Remarks and Acknowledgements*

It's impossible to discuss the evolution of this paper without discussing the close relationship between our work and work by D. Velleman which will appear in [14]. Accordingly, we shall attempt to discuss in parallel and in chronological order the evolution and mutual influences of these papers. This has the price of introducing an appearance of vagueness. For example the definitions of $\mathcal{S}_\mu$ and S_μ have evolved since we began work on this paper. Thus, at different stages of this evolution, Theorem 1 has had not only differing statements but differing contents. Nevertheless at each stage there was a clear analogue of Theorem 1. Accordingly, we shall refer to "something like Theorem 1" or "the analogue of the left-to-right implication of Theorem 1", etc. ...

This said, this paper grew out of the meeting of the two authors in Cambridge in 1978. At that time, Shelah had formulated a tentative version of $\mathcal{S}_\mu$ and S_μ and suggested to Stanley the possibility of proving something like the right to left implication of Theorem 1. Shelah felt that this was essentially just a warm-up for proving an analogous result with a stronger version of S_μ and morasses with built-in $\diamond$-principles (this material will appear in a sequel to this paper). Velleman, on the other hand, was interested from the beginning in finding a forcing principle *equivalent* to morasses without additional extra structure, and, influenced by our work, came later to morasses with built-in $\diamond$-principles. In the early stages both we and Velleman focused essentially on the special case $\mu = \aleph_1$ (though very early on Shelah envisaged applications for, say, inaccessible μ). Indeed, one of the main problems was to find the correct generalization of $\aleph_1$-closed, $\aleph_1$-closed having a certain number of "accidental" properties, and applying "accidentally" to certain partial orderings whose generalizations to μ are not even μ-closed.

Stanley and Velleman become aware of each other's work and established contact in December 1978. At about the same time, Shelah, drawing on earlier work of Laver [9] and Devlin [4] on Souslin trees with ascent paths, hit upon the notion of super-Souslin trees, the conditions for adding them of §2, and the general approach to proving Theorem 2, modulo something like the right-to-left implication of Theorem 1. This provided Stanley with the impetus to prove, in February 1979, a statement which evolved into the right-to-left implication of Theorem 1 in the special case $\mu = \aleph_1$. Shortly thereafter the authors met again:

Stanley provided the missing piece of the argument for Theorem 2, confirming that for $A \subseteq \omega_1$, there is an $(\aleph_1, 1)$-morass in L [A]. The authors turned to the question of something like the left-to-right implication of Theorem 1. Stanley hit upon the possibility of applying the principle analogous to $S_{\aleph_1}$ first to the countable conditions for forcing $\square_{\aleph_1}$ (see §4, Example 1), to obtain a $\square_{\aleph_1}$-sequence C, and then to the Jensen partial-order for obtaining an $(\aleph_1, 1)$-morass from C (see §3 of [11]). It was at this time that Shelah first realized the possibility of applying some version of a generalized Martin's Axiom to these orderings. This led to the results of [11], which were first announced and circulated with preliminary versions of this paper.

By Spring 1979, Velleman was able to prove that the existence of an $(\aleph_1, 1)$-morass is equivalent to a certain forcing principle for $\aleph_1$. We realized that the right-to-left implication of Theorem 1 would generalize to arbitrary regular $\mu > \aleph_1$ provided that the closure property of the analogue of S_μ was μ-closed with least upper bounds, and Velleman realized that his proof for $\aleph_1$ yielded the analogous generalization.

More importantly, in proving that his principle implies the existence of $(\aleph_1, 1)$-morasses, Velleman introduced a new partial order for adding morasses. This is essentially the order presented in §5; we thank Velleman for permitting us to present this here and in [12], [13]. The introduction of this order was essential for generalizing the left-to-right implication of Theorem 1 to higher cardinals. Stanley found the correct generalization in Autumn 1979, when he realized that (a suitable modification of) Velleman's order for μ was μ-directed closed, and that the proof of the right-to-left implication of Theorem 1 for μ goes through for μ-directed-closed orders. At the same time, Stanley realized that, by a theorem of Laver [8], this meant that supercompact cardinals can carry morasses; for this, and related developments, see [13].

The paper took its present form in January 1980, when Shelah, elaborating on an earlier idea, introduced the notion of uniform dense sets and ideals meeting them uniformly, thus permitting a more streamlined presentation of $\mathscr{S}_\mu$. Finally, the paper underwent one last transformation in March 1981. These changes affect §3 and §6 and are touched upon there.

In addition to the specific points cited above, we're indebted to Velleman and, by transitivity, his thesis advisor Kunen, for spotting many imprecisions and oversights, for suggesting improvements, and more generally for many helpful remarks. Laver first suggested to Stanley the possibility of forcing $\square_{\aleph_1}$ with countable conditions: in addition to this he was a source of many helpful suggestions and criticisms.

(1.3) *Morasses*

For the record, we give a complete definition of $(\mu, 1)$-morass for regular $\mu \geqq \aleph_1$. Nevertheless, the reader who has no or little prior knowledge of morasses, and who wishes to follow the arguments of §6 should consult [3], or [12] for a less compact, more instructive definition.

DEFINITION. Let $\mu \geqq \aleph_1$ be regular. A $(\mu, 1)$-morass is a structure

$$\mathcal{M} = (\mathcal{S}, S^0, S^1, \prec, \pi_{\bar\nu\nu})_{\bar\nu \prec \nu}$$

with the following properties:

(M$_0$): (a) $\mathcal{S}$ is a set of ordered pairs of primitively-recursively closed ordinals such that if $(\alpha, \nu) \in \mathcal{S}$, then $\alpha < \nu < \mu^+$,
(b) if (α, ν), $(\alpha', \nu') \in \mathcal{S}$ and $\alpha < \alpha'$ then $\nu < \alpha'$,
(c) $S^0 = \{\alpha : (\exists \nu)((\alpha, \nu) \in \mathcal{S})\}$, $S^1 = \{\nu : (\exists \alpha)((\alpha, \nu) \in \mathcal{S})\}$; for $\alpha \in S^0$, let $S_\alpha = \{\nu : (\alpha, \nu) \in \mathcal{S}\}$ (thus, by (a), (b), $(S_\alpha : \alpha \in S^0)$ is a partition of S^1); $\mu = \max S^0 = \sup S^0 \cap \mu$; $\mu^+ = \sup S^1 = \sup S_\mu$,
(d) for $\alpha \in S^0$, S_α is closed as a subset of $\sup S_\alpha$.

(M1): (a) $\prec$ is a tree on S^1 such that if $\bar\nu \prec \nu$ then $\alpha_{\bar\nu} < \alpha_\nu$,
(b) $(\pi_{\bar\nu\nu} : \bar\nu \prec \nu)$ is a commutative system of increasing maps, $\pi_{\bar\nu\nu} : \bar\nu \to \nu$,
(c) $\pi_{\bar\nu\nu} \restriction \alpha_{\bar\nu} = \mathrm{id} \restriction \alpha_{\bar\nu}$, $\pi_{\bar\nu\nu}(\alpha_{\bar\nu}) = \alpha_\nu$,
(d) $\pi_{\bar\nu\nu} : S_{\alpha_{\bar\nu}} \to S_{\alpha_\nu}$; $\bar\nu$ is minimal in $S_{\alpha_{\bar\nu}}$ iff ν is minimal in S_{α_ν}; $\bar\nu$ is a limit in $S_{\alpha_{\bar\nu}}$ iff ν is a limit in S_{α_ν}; if $\bar\nu$ immediately succeeds $\bar\tau$ in $S_{\alpha_{\bar\nu}}$, then ν immediately succeeds $\pi_{\bar\nu\nu}(\bar\tau)$ in S_{α_ν}.

(M2): if $\bar\nu \prec \nu$, $\bar\tau \in S_{\alpha_{\bar\nu}}$, $\tau = \pi_{\bar\nu\nu}(\bar\tau)$, then $\bar\tau \prec \tau$ and $\pi_{\bar\tau\tau} = \pi_{\bar\nu\nu} \restriction \bar\tau$.

(M3): $\{\alpha_{\bar\nu} : \bar\nu \prec \nu\}$ is closed in α_ν.

(M4): If ν is not maximal in S_{α_ν}, then $\{\alpha_{\bar\nu} : \bar\nu \prec \nu\}$ is unbounded in α_ν.

(M5): If $\{\alpha_{\bar\nu} : \bar\nu \prec \nu\}$ is unbounded in α_ν, then $\nu = \bigcup_{\bar\nu \prec \nu} \operatorname{range} \pi_{\bar\nu\nu}$.

(M6): If $\bar\nu \prec \nu$, $\bar\nu$, ν are limits in $S_{\alpha_{\bar\nu}}$, S_{α_ν} respectively, and if $\lambda = \sup \operatorname{range} \pi_{\bar\nu\nu} < \nu$, then $\bar\nu \prec \lambda$ and $\pi_{\bar\nu\lambda} = \pi_{\bar\nu\nu}$ (and so $\lambda \in S_{\alpha_\nu}$, by (M1)(c)).

(M7): Suppose $\bar\nu \prec \nu$, $\bar\nu$, ν are limits in $S_{\alpha_{\bar\nu}}$, S_{α_ν} respectively, and suppose range $\pi_{\bar\nu\nu}$ is cofinal in ν. Suppose further that $\bar\nu$ is the immediate $\prec$-predecessor of ν. Then there is no $\alpha \in S^0$ with $\alpha_{\bar\nu} < \alpha < \alpha_\nu$ such that for all $\bar\tau \in S_{\alpha_{\bar\nu}} \cap \bar\nu$, there is $\tau' \in S_\alpha$ with $\tau \prec \tau \prec \pi_{\bar\nu\nu}(\bar\tau)$.

§2. Super-Souslin trees

In this section we present a new class of trees, the super-Souslin trees, introduced by Shelah who was motivated by earlier work of Laver [9] and Devlin [4] on Souslin trees with ascent paths. Super-Souslin trees are characterized by a rather absolute positive property, and they have Souslin subtrees.

Next, we shall present Shelah's conditions for adding such trees. In addition to their interest in their own right, these conditions will serve to motivate the general concept of S-forcing introduced in the next section. This is why we shall take the time to develop in some detail the properties of these conditions and prove results which give more information than really needed if we were merely interested in forcing with this partial order. Finally, modulo the right to left implication of Theorem 1, we prove Theorem 2 in (2.16).

Throughout this section, κ will be an infinite cardinal.

(2.1) Let $\boldsymbol{T}=(T,\leqq)$ be a tree, $\lambda=\operatorname{ht}(\boldsymbol{T})$. $\boldsymbol{a}=(a_i : i<\lg \boldsymbol{a})$ is a *level sequence* from $\boldsymbol{T}$ iff $\boldsymbol{a}$ is 1-1 and for some $\alpha<\lambda$, for all $i<\lg \boldsymbol{a}$, $a_i\in T_\alpha$; in this case, we say $\boldsymbol{a}\in T_\alpha$ and $\alpha=|\boldsymbol{a}|$, by abuse of notation. $\operatorname{Lev}_\beta(\boldsymbol{T})$ is the set of level sequences $\boldsymbol{a}$ from $\boldsymbol{T}$ with $\lg \boldsymbol{a}=\beta$. If $\boldsymbol{a}$, $\boldsymbol{b}$ are level sequences from $\boldsymbol{T}$, set $\boldsymbol{a} \rightarrowtail \boldsymbol{b}$ iff $\lg \boldsymbol{a}=\lg \boldsymbol{b}$ and for all $i<\lg \boldsymbol{a}$, $a_i\leqq b_i$. Let $[\operatorname{Lev}_\beta(\boldsymbol{T})]^2=\{(\boldsymbol{a},\boldsymbol{b})\in \operatorname{Lev}_\beta(\boldsymbol{T}) : \boldsymbol{a}\rightarrowtail \boldsymbol{b}\}$.

(2.2) Definition. A κ^{++}-super-Souslin tree is a normal $\boldsymbol{T}=(T,\leqq)$ of height κ^{++} such that there is $F:[\operatorname{Lev}_\kappa(\boldsymbol{T})]^2\to\kappa^+$ satisfying

(*): if $F(\boldsymbol{a},\boldsymbol{b})=F(\boldsymbol{a},\boldsymbol{c})$ there is $i<\kappa$ such that b_i, c_i are comparable.

(2.3) Proposition. *If T is κ^{++}-super-Souslin, T has a κ^{++}-Souslin subtree which is also κ^{++}-super-Souslin.*

Proof. At several points we'll use that $\boldsymbol{T}$ is normal.

First note that if $\boldsymbol{T}$ is κ^{++}-super-Souslin, $x\in T$, and $\boldsymbol{T}_x$ is the restriction of $\boldsymbol{T}$ to $\{y\in T : x\leqq_T y\}$, then $\boldsymbol{T}_x$ is κ^{++}-super-Souslin.

Next, we prove:

(*): If $\boldsymbol{T}$ is κ^{++}-super-Souslin, if $A\subseteq T$ is an antichain of power κ^{++}, and if $Z=\{x\in T : \operatorname{card}(A\cap T_x)=\kappa^{++}\}$, then Z has no antichain of power κ.

Proof of (*). If Z has an antichain of power κ, then Z has a level sequence of length κ, $\boldsymbol{a}=(a_\xi : \xi<\kappa)$. Define by induction on $\alpha<\kappa^{++}$, $\boldsymbol{x}^\alpha=(x^\alpha_\xi : \xi<\kappa)$, $\boldsymbol{b}^\alpha=(b^\alpha_\xi : \xi<\kappa)$ such that for $\alpha<\kappa^{++}$, $\xi<\kappa$, $a_\xi <_T x^\alpha_\xi <_T b^\alpha_\xi$, $|b^\alpha_\xi|<|x^{\alpha+1}_\xi|$, $\boldsymbol{b}^\alpha$ a level sequence, $x^\alpha_\xi\in A$.

This suffices for the sought-after contradiction, since then there is $B \subseteq \kappa^{++}$ of power κ^{++} such that for $\alpha, \beta \in B$, $F(\boldsymbol{a}, \boldsymbol{b}^{\alpha}) = F(\boldsymbol{a}, \boldsymbol{b}^{\beta})$, so that, if $\alpha < \beta$, there is $\xi < \kappa$ such that $b^{\alpha}_{\xi} <_T b^{\beta}_{\xi}$. But then $x^{\alpha}_{\xi} <_T x^{\beta}_{\xi}$. The inductive construction can clearly be carried out, by cardinality considerations.

Now let $\boldsymbol{T}$ be κ^{++}-super-Souslin and let $\boldsymbol{a} = (a_i : i < \kappa)$ be any antichain of cardinal κ from $\boldsymbol{T}$. For $i < \kappa$, let $\boldsymbol{T}_i = \boldsymbol{T}_{a_i}$. If none of the $\boldsymbol{T}_i$ is κ^{++}-Souslin, there is $(A_i : i < \kappa)$ such that each A_i is an antichain of power κ^{++} in $\boldsymbol{T}_i$. But then $A = \bigcup_{i<\kappa} A_i$ is an antichain of $\boldsymbol{T}$ and $\{a_i : i < \kappa\} \subseteq Z$ (as above), which, by (*), is impossible. So some $\boldsymbol{T}_i$ is simultaneously κ^{++}-super-Souslin and κ^{++}-Souslin. Note that what we've really proved is: suppose $A \subseteq T$ is an antichain. Then

$$\{a \in A : \boldsymbol{T}_a \text{ is } \textit{not } \kappa^{++}\text{-Souslin}\} \text{ has cardinal} < \kappa.$$

(2.4) LEMMA. *Suppose* $V' \subseteq V$, $V' \models \mathrm{ZFC}$, $\mathbf{P}(\kappa) \subseteq V'$, $(\kappa^{++})^{V'} = \kappa^{++}$. *If* $\boldsymbol{T} \in V'$ *and* $V' \models$ "$\boldsymbol{T}$' *is* κ^{++}*-super-Souslin" then* $\boldsymbol{T}$ *is* κ^{++}*-super-Souslin.*

PROOF. By the absoluteness properties assumed for V', a function F which witnesses super-Souslinity in V' witnesses super-Souslinity in V.

(2.4) thus gives the "raison-d'être" of κ^{++}-super-Souslin trees: they stay super-Souslin unless κ^{++} is collapsed or new subsets of κ are introduced. We now present the conditions for adding a κ^{++}-super-Souslin tree.

(2.5) DEFINITION. Let $p \in P$ iff $p = (x, \boldsymbol{t}, f)$ where:

(i) $x \in [\kappa^{++}]^{<\kappa^+}$; if $\alpha > 0$ and $\alpha \in x$, then $\alpha \in \mathrm{Lim}$ iff o.t. $(x \cap \alpha) \in \mathrm{Lim}$, and if $\alpha \in x$ is a successor, then the predecessor of α is in x,

(ii) Let $\theta = \text{o.t. } x$ and let $(\alpha_i : i < \theta)$ increasingly enumerate x; $\boldsymbol{t} = (t, \leqq)$ is a κ-normal tree (i.e., distinct points on limit levels have distinct sets of predecessors and all points have κ distinct successors on all higher levels) of $\mathrm{ht}(\boldsymbol{t}) = \theta$, and for $i < \theta$, t_i is a proper initial segment of $[\kappa^+ . \alpha_i, \kappa^+ . (\alpha_i + 1))$,

(iii) f is a function, $\mathrm{card} f \leqq \kappa$, $f : \mathrm{dom} f \to \kappa^+$, $\mathrm{dom} f \subseteq [\mathrm{Lev}_\kappa(\boldsymbol{t})]^2$, and (*) of (2.2) holds, replacing "F" by "f", and requiring that there be κ many i such that b_i, c_i are comparable.

$$(x, \boldsymbol{t}, f) \geqq (x', \boldsymbol{t}', f') \quad \text{iff } x \supseteq x', \quad \boldsymbol{t}' \text{ is a subtree of } \boldsymbol{t}, \quad f \supseteq f'.$$

Set $\mathbf{P} = (P, \geqq)$.

(2.6) LEMMA. $\mathbf{P}$ *is* κ^+*-closed, with least upper bounds.*

PROOF. Clear, since the triple of unions of the co-ordinates is the lub.

(2.7) PROPOSITION. *If* $(x', \boldsymbol{t}', f') \in P$, $(\boldsymbol{a}, \boldsymbol{b}) \in [\mathrm{Lev}_\kappa(\boldsymbol{t})]^2 \setminus \mathrm{dom} f$, *then there is* $(x', \boldsymbol{t}', f) \geqq (x', \boldsymbol{t}', f')$ *with* $(\boldsymbol{a}, \boldsymbol{b}) \in \mathrm{dom} f'$ *and* $f(\boldsymbol{a}, \boldsymbol{b}) \notin \mathrm{range} f'$.

PROOF. Clear.

The next propositions guarantee that conditions can be extended to add arbitrary levels and arbitrary points on an already present level. The proofs are obvious and will be omitted, except to say that the κ-branching trees of (2.8) are used in the proofs of (2.10), (2.11) (as well as in (2.14)), and that in (2.11) (adding levels), sometimes more than ξ must be added to x' in order to respect the last clauses of (2.5)(i).

(2.8) DEFINITION. A κ-normal tree t is *κ-branching* if, letting $\theta = \mathrm{ht}(t)$, there is a 1-1 function $(b^x : x \in t)$ such that for $x \in t$, $x \in b^x$ is a branch of height θ. Note that if t is κ-branching then for all x not on the top level there are at least κ branches of height θ through x.

(2.9) PROPOSITION. *If $(x, t, f) \in P$, there's κ-branching t' such that $(x, t', f) \geqq (x, t, f)$.*

(2.10) PROPOSITION. *If $(x, t, f) \in P$, if $\nu < \kappa^{++}$, if $\kappa^+ . \alpha$ is the largest multiple of κ which is $\leqq \nu$, and if $\alpha \in x$, then there is $(x, t', f) \geqq (x, t, f)$ with $\nu \in t'$.*

(2.11) PROPOSITION. *If $(x, t, f) \in P$, $\xi \in \kappa^{++}$, there is $(x', t', f) \geqq (x, t, f)$ with $\xi \in x'$.*

(2.12) We now begin to develop certain properties of $\mathbf{P}$ which will help to establish the κ^{++}-c.c., assuming $2^\kappa = \kappa^+$, and, as mentioned above, serve to motivate the general notion of S-forcing. We shall define "contractions" $\bar{p}$ of elements $p \in P$. The set of contractions will have power κ^+, assuming $2^\kappa = \kappa^+$. Two elements p_1, p_2 will have the same contraction just in case, roughly speaking, p_1 and p_2 have the same underlying structure, and differ only in that, letting $p_i = (x_i, t^i, f_i)$, the "sets of indices" x_1, x_2 are different (though part of the meaning of having the same underlying structure will be that o.t. $x_1 =$ o.t. x_2).

So, let $p = (x, t, f) \in P$. Let $s = (\alpha_i : i < \theta)$ increasingly enumerate x. Let $\bar{x} = \theta$. Define σ_s on $\bigcup_{i<\theta} [\kappa^+ . \alpha_i, \kappa^+ . (\alpha_i + 1))$ by:

$$\sigma_s((\kappa^+ . \alpha_i) + \eta) = (\kappa^+ . i) + \eta.$$

Let σ_s^* be the extension of σ_s to ${}^\kappa[\mathrm{dom}\, \sigma_s]$ by pointwise images, i.e. $\sigma_s^*((\xi_i : i < \kappa)) = (\sigma_s(\xi_i) : i < \kappa)$. Let $\bar{t}$ be the normal tree isomorphic to t by σ_s. Let $\bar{f} = f \circ (\sigma_s^*)^{-1}$, i.e. $\bar{f}(\sigma_s^*(\boldsymbol{a}), \sigma_s^*(\boldsymbol{b})) = f(\boldsymbol{a}, \boldsymbol{b})$. Then $\bar{p} = (\bar{x}, \bar{t}, \bar{f})$ is the contraction of p. Since $\bar{x} = \theta < \kappa^+$, set $\lg \bar{p} = \theta$ and note that $\bar{t} \in [\kappa^+ . (\theta + 1)]^{<\kappa^+}$. Then it is clear that if $2^\kappa = \kappa^+$ then $\mathcal{T} = \{\bar{p} : p \in P\}$ has power κ^+.

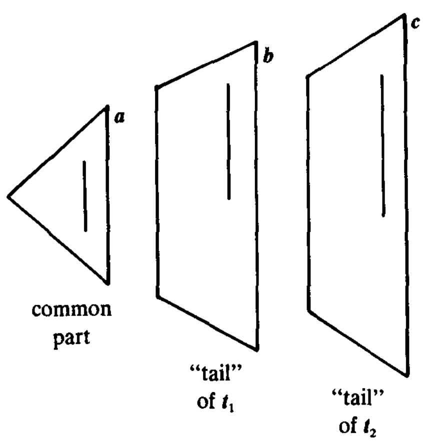
(2.13) Our goal is to show that **P** has the κ^{++}-c.c., assuming $2^\kappa = \kappa^+$. Once this is done it is clear that in $V^{\mathbf{P}}$ there is a κ^{++}-super-Souslin tree. We will prove a very strong form of κ^{++}-c.c. in the following way: we will show

(*) (Amalgamation Property) If $p_1 = (x_1, t^1, f_1)$, $p_2 = (x_2, t^2, f_2) \in P$, if $\bar{p}_1 = \bar{p}_2$ and x_1, x_2 have the strong Δ-property, then p_1, p_2 are compatible.

First, let's see that if $2^\kappa = \kappa^+$ this gives κ^{++}-c.c. Suppose $A \in [P]^{\kappa^{++}}$. Then there is $B \in [A]^{\kappa^{++}}$ such that all elements of B have the same contraction and $\{x^p : p \in B\}$ forms a strong Δ-system, where for $p \in P$, $p = (x^p, t^p, f^p)$. But then by (*), the members of B are pairwise compatible and hence A was not an antichain.

(2.14) We now turn to the proof of (*) of (2.13). Suppose p_1, p_2 are as in (*). Let $\bar{p}_1 = \bar{p}_2 = (\theta, \bar{t}, \bar{f})$ and let $s^i = (\alpha^i_j : j < \theta)$ increasingly enumerate x_i $(i = 1, 2)$. Let $j_0 < \theta$ be such that $s^1 \restriction j_0 = s^2 \restriction j_0$, but (say) $\alpha^1_j < \alpha^2_{j_0}$ for $j_0 \leqq j < \theta$. By (2.9) we may clearly suppose that $\bar{t}$ and hence t^1, t^2 are κ-branching (in fact, only t^1 need be κ-branching and only if $\theta \in \mathrm{Lim}$).

The only problem is to respect (2.5) (iii) (and hence the "κ-many" version of (2.2) (*)). The typical situation which could arise is that we have $(\boldsymbol{a}, \boldsymbol{b}) \in [\mathrm{Lev}_\kappa(t^1)]^2$, $(\boldsymbol{a}, \boldsymbol{c}) \in [\mathrm{Lev}_\kappa(t^2)]^2$, $f_1(\boldsymbol{a}, \boldsymbol{b}) = f_2(\boldsymbol{a}, \boldsymbol{c})$, range $\boldsymbol{a} \subseteq [\kappa^+ . s^2_j, \kappa^+ . (s^1_j + 1))$, range $\boldsymbol{b} \subseteq [\kappa^+ . s^1_{k_1}, \kappa^+ . (s^1_{k_1} + 1))$, range $\boldsymbol{c} \subseteq [\kappa^+ . s^2_{k_2}, \kappa^+ . (s^2_{k_2} + 1))$ where $j < j_0$, $k_1, k_2 \geqq j_0$. Then, in weaving t^1, t^2 together, we must guarantee that for κ many ξ, b_ξ will precede c_ξ. The approach is to run a height θ branch through t^1 which contains b_ξ, and appoint this branch to be the set of predecessors of d_ξ where d_ξ is the t^2 predecessor of c_ξ on level j_0,



and to do this for κ-many ξ.

There are two things to notice first, to wit:

(1) $\boldsymbol{d}$ is 1-1 (i.e. for $\xi \neq \xi'$, $d_\xi \neq d_{\xi'}$): this is because, since $\boldsymbol{a} \dashv^2 \boldsymbol{c}$, and the level of $\boldsymbol{a}$ in $\boldsymbol{t}^2$ (i.e. j) is less than the level of $\boldsymbol{d}$ in $\boldsymbol{t}^2$ (i.e. j_0), we must have $\boldsymbol{a} \dashv^2 \boldsymbol{d}$ and so $\boldsymbol{d}$ is 1-1.

(2) For κ-many $\xi < \kappa$, the set of t^2 predecessors of $d_\xi \subseteq$ the set of t^1-predecessors of b_ξ. This is argued as follows: let $\boldsymbol{c}'$ be "the replica of $\boldsymbol{c}$" in t^1, i.e. for $\xi < \kappa$, $c'_\xi = \sigma_{s^1}(\sigma_{s^2}^{-1}(c_\xi))$. Then $f^1(\boldsymbol{a}, \boldsymbol{b}) = f^1(\boldsymbol{a}, \boldsymbol{c}')$, so that for κ-many $\xi < \kappa$, b_ξ, c'_ξ are comparable. Hence for κ-many $\xi < \kappa$, d'_ξ is the t'-predecessor, on t^1-level j_0, of b_ξ, where $\boldsymbol{d}'$ is the replica in t^1 of $\boldsymbol{d}$.

Finally, since there are only κ-many situations of the sort outlined above, enumerate them in type κ, with κ repetitions of each situation. By (1), (2) above, by the fact that at any stage we've chosen $< \kappa$ branches so far, and the fact that t^1 was assumed to be κ-branching, we can inductively choose ξ's and branches as desired, in a 1-1 manner.

This means that all requirements stemming from the need to make $f_1 \cup f_2$ satisfy (2.5) (iii) have been met. Now continue weaving together t^1, t^2 essentially arbitrarily. Add new points, and possibly new levels, and then rename points if necessary to guarantee that (2.5) (i), (ii) are satisfied. Call the resulting tree t and its set of levels x. Then $(x, t, f_1 \cup f_2) \geqq p_1, p_2$.

(2.15) (a) Suppose $\tau = (\theta, \bar{t}, \bar{f}) \in \mathcal{T}$, and $s : \theta \to \kappa^{++}$ is increasing. τ "acts on" s naturally to give $p = (x, t, f) \in P$, with $x = \text{range}\, s$ and $\bar{p} = \tau$ (merely reverse the construction in (2.12)); p will be denoted by $\tau(s)$.

(b) Suppose $p = \tau(s) \geqq q = \tau'(s')$. Then, clearly we must have range $s \supseteq$ range s'; i.e. there is a unique increasing $g : \lg s' \to \lg s$ such that for $i < \lg s'$, $s'_i = s_{g(i)}$.

(c) (Indiscernibility Property) Suppose $\tau(s^1)$, $\tau(s^2)$, $\hat{\tau}(\hat{s}^1)$, $\hat{\tau}(\hat{s}^2) \in P$, and suppose that $\tau(s^1) \geqq \hat{\tau}(\hat{s}^1)$. Let $g : \lg \hat{s}^1 \to \lg s^1$ be as guaranteed by (b). It is then easy to verify that if:

(*): for all $i < \lg \hat{s}^2 (= \lg \hat{s}^1)$, $\hat{s}^2_i = s^2_{g(i)}$,

then

(**): $\tau(s^2) \geqq \hat{\tau}(\hat{s}^2)$.

This says that from the point of view of τ, $\hat{\tau}$, the pair s^1, $\hat{s}^1$ is indiscernible from the pair s^2, $\hat{s}^2$, provided that the co-ordinates of $\hat{s}^2$ are distributed among the co-ordinates of s^2 in the same way that the co-ordinates of $\hat{s}^1$ are distributed among the co-ordinates of s^1.

(d) (Sufficiently Generic Sets) Suppose D is a dense subset of $\mathcal{T}$ (with the ordering $\geqq \mid \mathcal{T}$). Set $p \in D^{\#}$ iff $p = \tau(s)$ for some $\tau \in D$ and increasing $s : \lg \tau \to \kappa^{++}$. A subset of P is uniform dense just in case it is $D^{\#}$ for some dense $D \subseteq \mathcal{T}$.

If D is a dense subset of $\mathcal{T}$ and G is an ideal in $\mathbf{P}$ then G meets $D^{\#}$ uniformly iff

(*) $(\forall s \in [\kappa^{++}]^{<\kappa^+})(\exists \tau \in D)(\exists s' \in [\kappa^{++}]^{<\kappa^+})$

$$[\text{range}\, s \subset \text{range}\, s' \text{ and } \lg s' = \tau \text{ and } \tau(s') \in G].$$

Note that if G meets $D^{\#}$ uniformly, then G meets all the $D_\alpha = \{\tau(s) \in P : \alpha \in \text{range}\, s\}$.

For $\alpha < \kappa^{++}$, let $I_\alpha = [\kappa^+ . \alpha, \kappa^+ . (\alpha + 1))$, and if $i < \kappa^+$, let $I^i_\alpha = [\kappa^+ . \alpha, (\kappa^+ . \alpha) + i)$. Assuming $2^\kappa = \kappa^+$, let $((g_i, h_i) : i < \kappa^+)$ enumerate $\{(g, h) : g, h : \kappa \to \kappa^+ \text{ are 1-1}\}$. If $\alpha < \kappa^+$, let $\boldsymbol{x}^{i,\alpha}$, $\boldsymbol{y}^{i,\alpha}$ be the κ-sequences from I_α defined by: $x^{i,\alpha}_\xi = (\kappa^+ . \alpha) + g_i(\xi)$, $y^{i,\alpha}_\xi = (\kappa^+ . \alpha) + h_i(\xi)$.

For $i < \kappa^+$, if $\tau = (\theta, \bar{t}, \bar{f}) \in \mathcal{T}$, set $\tau \in D_i$ iff for all $\eta < \theta$, $I^i_\eta \subseteq \bar{t}$ and if $\eta < \eta' < \theta$, then range $\boldsymbol{x}^{i,\eta} \cup$ range $\boldsymbol{y}^{i,\eta'} \subseteq \bar{t}$, and if $(\boldsymbol{x}^{i,\eta}, \boldsymbol{y}^{i,\eta'}) \in [\text{Lev}_\kappa(\bar{t})]^2$, then $(\boldsymbol{x}^{i,\eta}, \boldsymbol{y}^{i,\eta'}) \in \text{dom}\, \bar{f}$. Thus, in virtue of (2.6), (2.7), (2.10), (2.11), each D_i is a dense subset of $\mathcal{T}$.

Assuming $2^\kappa = \kappa^+$, G is *sufficiently generic* if G meets all the $D^{\#}_i$ uniformly, $i < \kappa^+$.

The justification for this terminology is that, if G is sufficiently generic and if $\boldsymbol{T} = \bigcup_{p \in G} \boldsymbol{t}^p$, $F = \bigcup_{p \in G} f^p$, then $\boldsymbol{T}$ is κ^{++}-super-Souslin with F as witness. This is argued as follows where, for $p \in P$, $p = (x^p, \boldsymbol{t}^p, f^p)$.

Clearly $\boldsymbol{T}$ is a normal tree of height κ^{++} and for $\alpha < \kappa^{++}$. $T_\alpha = I_\alpha$. We must see that if $(\boldsymbol{a}, \boldsymbol{b}) \in [\text{Lev}_\kappa(\boldsymbol{T})]^2$ then $(\boldsymbol{a}, \boldsymbol{b}) \in \text{dom}\, F$. So, let $\alpha < \beta < \kappa^{++}$, $i < \kappa^+$ be such that $\boldsymbol{a} = \boldsymbol{x}^{i,\alpha}$, $\boldsymbol{b} = \boldsymbol{y}^{i,\beta}$, choose $s \in [\kappa^{++}]^{<\kappa^+}$ with $\alpha, \beta \in$ range s, and let $\tau \in \mathcal{T}$, $s' \in [\kappa^{++}]^{<\kappa^+}$ be such that $\tau \in D_i$, $\tau(s') \in G$, range $s \subseteq$ range s'. Let $p = \tau(s')$. Clearly $\boldsymbol{a}$, $\boldsymbol{b}$ are level sequences from $\boldsymbol{t}^p$. Also, since $(\boldsymbol{a}, \boldsymbol{b}) \in [\text{Lev}_\kappa(\boldsymbol{T})]^2$, we must have that $(\boldsymbol{a}, \boldsymbol{b}) \in [\text{Lev}_\kappa(\boldsymbol{t}^p)]^2$ (since otherwise for some $\xi < \kappa$, a_ξ is not the predecessor on level α of b_ξ in $\boldsymbol{t}^p$, but then not in $\boldsymbol{T}$ either). But then, since $p = \tau(s')$, $\tau \in D_i$, by our choice of i, $(\boldsymbol{a}, \boldsymbol{b}) \in \text{dom}\, f^p \subseteq \text{dom}\, F$.

(2.16) It will be a consequence of the right to left implication of Theorem 1 that if $2^\kappa = \kappa^+$ and there's a $(\kappa^+, 1)$ morass then there is a sufficiently generic $G \subseteq P$, and hence that there is a κ^{++}-super-Souslin tree. Further, assuming $2^\kappa = \kappa^+$, and that κ^{++} is, in L, a successor cardinal, it is easy to find $A \subseteq \kappa^+$ such that $\mathcal{P}(\kappa) \subseteq \text{L}[A]$ (and hence $[\kappa^{++}]^{<\kappa^+} \subseteq \text{L}[A]$), and $\kappa^{++} = (\kappa^{++})^{\text{L}[A]}$. This argument permits us to prove:

THEOREM 2. $2^{\kappa} = \kappa^{+} \wedge \mathrm{SH}_{\kappa^{++}} \Rightarrow \kappa^{++}$ *is (inaccessible)*$^{\mathrm{L}}$.

PROOF. It is well-known that in L[A] (A as above), there is a $(\kappa^{+}, 1)$-morass (cf. [12]). By the absoluteness properties of L[A], a $(\kappa^{+}, 1)$-morass in L[A] is actually a $(\kappa^{+}, 1)$-morass in V (the morass properties are all absolute, modulo cardinality).

Hence there is a κ^{++}-super-Souslin tree (in V) which has a κ^{++}-Souslin subtree. In fact, if $\boldsymbol{T}$ is κ^{++}-super-Souslin in L[A], $\boldsymbol{T}$ is κ^{++}-super-Souslin.

§3. S-forcing

In this section, we present the general concept of S-forcing, of uniform dense sets and filters meeting them uniformly, appealing to the motivating example analyzed in §2. We prove a simple lemma about constructing conditions in S-forcing notions. This will serve in the proof of the right-to-left implication of Theorem 1. We also include some material which provides a translation between our principles S_{μ} and Velleman's forcing principles [14].

Our original treatment of S-forcing differed in the following sense. The Restriction Property (3.8), below, was assumed, and the partial order was required to be μ-directed closed.

In the course of providing the translation between our principles and Velleman's, Stanley realized that the Restriction Property could be dropped, but at the cost of a slight strengthening of μ-directed closure. He subsequently realized that in the presence of the other properties of S-forcing, the conjunction of the Restriction Property and μ-directed closure imply strong μ-directed closure (see (3.11), below). Accordingly, in the presentation below, the Restriction Property has been dropped and an S-forcing notion is required to be strongly-directed closed.

Further, in §6, the proof of the right-to-left implication of Theorem 1 is a modification of our original proof, when we were assuming the Restriction Property, and incorporates an idea coming from Velleman's proof [14] of the equivalence between the existence of morasses and his forcing principles. We thank Velleman for permitting us to use this material here.

(3.1) In what follows, we shall consider forcing notions $\mathbf{P} = (P, \geqq)$ which we shall call *μ-special* and which have the following properties:

(a) The elements of P have the form $\tau(s)$ where $\tau \in \mathcal{T}$ is a "term", and s is an increasing sequence from μ^{+} of length $\lg \tau$, an ordinal depending on τ, with $\lg \tau < \mu$; further, all of the $\tau(s)$ ($\tau \in \mathcal{T}$, $s \in [\mu^{+}]^{\lg \tau}$) are elements of $\mathbf{P}$. (*Remark*:

if we're actually going to force with **P** it's natural to require that card $\mathcal{T} = \mu$; this usually amounts to assuming $2^{<\mu} = \mu$. However, this is not necessary for the principle S_μ. On the other hand, as in §2, we may need $2^{<\mu} = \mu$ to show that the required collection of uniform dense sets has small cardinality.) More succintly, viewing $\tau(s)$ as the pair (τ, s), there is a set $\mathcal{T}$ and a function lg,

$$\lg : \mathcal{T} \to \mu \quad \text{such that } P = \bigcup_{\tau \in \mathcal{T}} \{\tau\} \times [\mu^+]^{\lg \tau}.$$

Further, exactly one term $O_{\mathcal{T}}$ has length 0; $O_{\mathcal{T}}(\varnothing)$ will be denoted by $O_{\mathbf{P}}$ and will be the trivial condition. Intuitively, the terms τ "operate on" all increasing sequences from μ^+ of a certain length, $\lg \tau < \mu$, appropriate for τ, to give the elements of P. In §2, the terms are the "contractions" of conditions (cf. (2.12), (2.15) (a)), the length, $\lg \tau$, of τ is θ, where $\tau = (\theta, \bar{t}, \bar{f})$. In the general setting as well, the term τ may be thought of as $\tau\ (\mathrm{id} \mid \lg \tau)$ and hence as lying in P. We'll write $\tau' \geqq \tau$ for $\tau'(\mathrm{id} \mid \lg(\tau')) \geqq \tau(\mathrm{id} \mid \lg \tau)$. Also, for $p \in P$, we'll write $p = \tau^p(s^p)$.

(b) $\tau'(s') \geqq \tau(s) \Rightarrow \text{range } s' \supseteq \text{range } s$.

Note that in §2, κ^+ plays the role of μ.

(3.2) Remark (corresponds to (2.15)(c)). If **P** is μ-special, $q = \tau(s)$, $p = \tau'(s') \in P$ and $p \geqq q$ then there is a unique order-preserving $g : \lg \tau \to \lg \tau'$ (recall that $\lg \tau = \lg s$, $\lg \tau' = \lg s'$), such that for $i < \lg \tau$, $s_i = s'_{g(i)}$. This is clear by (3.1)(b).

(3.3) Uniform Dense Sets

Suppose **P** is μ-special, $D \subseteq \mathcal{T}$ is dense. Set $p \in D^{\#}$ iff $p = \tau(s)$ for some $\tau \in D$, and increasing $s : \lg \tau \to \mu^+$. A dense subset of **P** is uniform if it is $D^{\#}$ for some dense $D \subseteq \mathcal{T}$.

If $D \subseteq \mathcal{T}$ is dense and G is an ideal in **P** then G meets $D^{\#}$ uniformly if

$$(*)\ (\forall s \in [\mu^+]^{<\mu})(\exists \tau \in D)(\exists s' \in [\mu^+]^{<\mu})$$

$$[\text{range } s \subseteq \text{range } s' \text{ and } \lg s' = \lg \tau \text{ and } \tau(s') \in G].$$

Note that if G meets $D^{\#}$ uniformly then G meets all of the $D_\alpha = \{\tau(s) \in P : \alpha \in \text{range } s\}$, $\alpha < \mu^+$.

For the remainder of this section all **P** *are μ-special.*

(3.4) Definition. **P** satisfies the *Indiscernibility Property* (cf. (2.15)(d)) iff whenever $\tau(s^1)$, $\tau(s^2)$, $\hat{\tau}(\hat{s}^1)$, $\hat{\tau}(\hat{s}^2) \in \mathbf{P}$, if $\tau(s^1) \geqq \hat{\tau}(\hat{s}^1)$, if $g : \lg \hat{\tau} \to \lg \tau$ is as guaranteed by (3.2) for $\tau(s^1)$, $\hat{\tau}(\hat{s}^1)$, and if:

(*) for all $i < \lg \hat{\tau}$, $\hat{s}^2_i = s^2_{g(i)}$ (i.e., if the same g works), then:

(**): $\tau(s^2) \geqq \hat{\tau}(\hat{s}^2)$.

(3.5) Remark. If **P** satisfies Indiscernibility, then **P** is completely determined by: (a) the function $\lg : \mathcal{T} \to \mu$, (b) a function K with $\operatorname{dom} K = \mathcal{T} \times \mathcal{T}$ and $K(\tau, \tau')$ a (possibly empty) set of order-preserving functions from $\lg \tau$ to $\lg \tau'$ (K tells us all the possible functions g as in (3.2)).

(3.6) Definition. P satisfies the *Amalgamation Property* (cf. (2.13)(*)) iff whenever $\tau(s)$, $\tau(s') \in P$ and (s, s') has the strong Δ-property then $\tau(s)$, $\tau(s')$ are compatible ($\tau(s)$, $\tau(s')$ need not have a lub).

(3.7) Definition. P satisfies the *Extension Property* (cf. (2.11)) iff whenever $p = \tau(s) \in P$ and $\xi < \mu^+$, there is $\tau'(s') \geqq p$ with $\xi \in \operatorname{range} s'$.

(3.8) Definition. **P** satisfies the *Restriction Property* iff there is a function $r : \mathcal{T} \times \mu \to \mathcal{T}$ such that:

(a) if $\lg \tau \leqq \eta < \mu$, then $r(\tau, \eta) = \tau$,

(b) if $\eta < \lg \tau$, $\lg(r(\tau, \eta)) = \eta$, and for all $s \in [\mu^+]^{\lg \tau}$, $\tau(s) \geqq r(\tau, \eta)(s \mid \eta)$,

(c) $r(\tau, \eta') = r(r(\tau, \eta), \eta')$ for all $\eta, \eta' < \mu$,

(d) if $\tau(s) \geqq \bar{\tau}(\bar{s})$, let g be as guaranteed by (3.2); let $\bar{\eta} < \mu$, and let $\eta = \sup g''\bar{\eta}$; then $r(\tau, \eta)(s \mid \eta) \geqq r(\bar{\tau}, \bar{\eta})(\bar{s} \mid \bar{\eta})$.

(3.9) Definition. If $p \in P$, set $p = \tau^p(s^p)$. Suppose **P** is μ-special. Then **P** is *strongly μ-directed closed* iff whenever $D \in [P]^{<\mu}$ is directed, there is an upper bound p^* for D with $\operatorname{range}(s^{p^*}) = \bigcup_{p \in D} \operatorname{range}(s^p)$. Note that **P** of §2 is strongly μ-directed closed.

(3.10) Remark. Suppose **P** is μ-special and indiscernible.

(1) **P** is μ-directed closed iff **P** satisfies the apparently weaker property:

(*) whenever $\bar{D} \in [P]^{<\mu}$ is directed and for $\bar{p} \in \bar{D}$, $\operatorname{range}(s^{\bar{p}}) \subseteq \mu$, there's an upper bound for $\bar{D}$.

This is argued as follows. Let $D \in [P]^{<\mu}$ be directed, and let s^* be the increasing enumeration of $\bigcup_{p \in D} \operatorname{range}(s^p)$. Let $\theta = \lg s^*$. For $p \in D$, let $\bar{s}^p = (s^*)^{-1} \circ s^p$, so range $\bar{s}^p \subseteq \theta < \mu$. Further, $\bar{D} = \{\tau^p(\bar{s}^p) : p \in D\}$ is directed, by indiscernibility. So, $\bar{D}$ has an upper bound $\bar{p}^* = \tau'(\bar{s}')$ and range $\bar{s}' \supseteq \theta$. Let $\theta' = \text{o.t.}\, \bar{s}'$, and let s' be an increasing function with length θ' and such that $s' \mid \theta = s^*$. Then $p' = \tau'(s')$ is an upper bound for D, by indiscernibility, since $\bar{p}$ is an upper bound for $\bar{D}$.

The same argument shows that:

(2) **P** is strongly μ-directed closed iff

(**) whenever $\bar{D} \in [P]^{<\mu}$ is directed and for $\bar{p} \in \bar{D}$, range $s^{\bar{p}} \subseteq \mu$, there's an upper bound $\bar{p}^*$ for $\bar{D}$ with range $s^{\bar{p}^*} = \bigcup_{\bar{p} \in \bar{D}}$ range $s^{\bar{p}}$.

Further, the argument of (1) can be used to show:

(3) If **P** is μ-directed closed with the Restriction Property then **P** is strongly μ-directed closed.

The point is that the upper bound p' obtained in (1) has the property that range $s^{p'}$ is an end-extension of range s^*. Hence we can apply Restriction to conclude that $r(\tau', \theta)(s' \mid \theta) = r(\tau', \theta)(s^*)$ is an upper bound for D (by the monotonicity of the restriction operator).

(3.11) Definition. $\mathbf{P} \in \mathcal{S}_\mu$ (**P** is a μ-S-forcing notion) iff **P** is μ-special, satisfies Indiscernibility, Amalgamation, Extension and:

P is strongly μ-directed closed.

(3.12) Question. Give a Boolean algebraic characterization of $\{\mathbf{B}(\mathbf{P}) : \mathbf{P} \in \mathcal{S}_\mu\}$.

(3.13) Proposition. *Suppose* $\mathbf{P} \in \mathcal{S}_\mu$, $p = \tau(s) \in P$, range $s \subseteq \xi < \mu$. *Then there is* $\tau'(s') \geqq p$ *with* $s' \supseteq \mathrm{id} \mid \xi$.

Proof. Using Extension, and just κ-closure, it's easy to produce such $\tau'(s')$.

(3.14) *A Translation Between* S_μ *and Velleman's Forcing Principle for* μ

The basic notion in Velleman's forcing principle, [14], is that of an indiscernible family of dense open sets. So, let $\mu > \omega$ be regular, **P** a partial ordering, $\mathcal{D} = (D_\alpha : \alpha < \mu^+)$ a family of μ^+ dense open sets. For $p \in P$, let *realm of* $p = \mathrm{rlm}\, p = \{\alpha : p \in D_\alpha\}$. For $\alpha > \mu^+$, let $P_\alpha = \{p : \mathrm{rlm}\, p \subseteq \alpha\}$ and let $P^* = \bigcup_{\alpha < \mu} P_\alpha$. So $P^* \subseteq P_\mu$.

$\mathcal{D}$ is μ-*indiscernible* iff:

(I1) $P^* \neq \varnothing$ and $(\forall \alpha < \mu)(D_\alpha \cap P^*)$ is dense open in $\mathbf{P}^* = (P^*, \leqq)$.

(I2) $(\forall \alpha < \mu)\mathbf{P}_\alpha = (P_\alpha, \leqq)$ is μ-directed closed.

(I3) For all increasing $f : \alpha \to \gamma$ with $\alpha < \mu$, $\gamma < \mu^+$ there's order-preserving $\sigma_f : \mathbf{P}_\alpha \to \mathbf{P}_\gamma$, such that:

(I4) $(\forall p \in P_\alpha)(\mathrm{rlm}\, \sigma_f(p) = f''\, \mathrm{rlm}\, p)$.

(I5) (Amalgamation) If $\gamma < \mu$, if $\beta < \alpha$, $f \mid \beta = \mathrm{id} \mid \beta$, $f(\beta) \geqq \alpha$, then p and $\sigma_f(p)$ are compatible in $\mathbf{P}^*$.

(I6) (Commutativity) $(f_1:\alpha_1\to\alpha_2,\ f_2:\alpha_2\to\gamma,\ \alpha_1,\alpha_2<\mu,\ \gamma<\mu^+,\ f_1,\ f_2$ increasing) $\Rightarrow \sigma_{f_2\circ f_1}=\sigma_{f_2}\circ\sigma_{f_1}$.

Velleman introduces weakenings and strengthenings of this notion. The weakening is obtained by not requiring the D_α to be open dense (though still requiring (I1)), and by modifying (I3) so as to require the existence of σ_f only for nice functions f (which Velleman calls "SLOP" functions). In this case $(D_\alpha:\alpha<\mu^+)$ is called almost μ-indiscernible.

(3.14.1) The strengthening is obtained by imposing additional requirements (I7)–(I12); (I11), (I12) will not be important for us here, so we won't state them here:

(I7) If $D\in[P]^{<\mu}$ is directed then D has a lower bound p^* with $\operatorname{rlm} p^*=\bigcup_{p\in D}\operatorname{rlm} p$.

(I8) $(\forall p\in P)$ $\operatorname{card}\operatorname{rlm} p<\mu$ (so $P^*=P_\mu$).

(I9) If $f:\alpha\to\gamma$ is increasing, $\alpha<\mu$, $\gamma<\mu^+$ and $\sigma_f(p)\leqq\sigma_f(q)$ then $p\leqq q$.

(I10) If $p\in P$, there's unique q with $\operatorname{rlm} q=\alpha<\mu$ and increasing $f:\alpha\to\mu^+$ such that $p=\sigma_f(q)$.

(3.14.2) PROPOSITION. (a) $(\mathbf{P},\mathscr{D})$ *satisfies* (I1), (I3)–(I6), (I8)–(I10) *iff* $\mathbf{P}$ *is μ-special, Indiscernible satisfying Extension and Amalgamation, and for* $\alpha<\mu^+$, $D_\alpha=\{p:\alpha\in\operatorname{range} s^p\}$.

(b) *Further, in the presence of* (I1), (I3)–(I6), (I8)–(I10), $(\mathbf{P},\mathscr{D})$ *satisfies* (I2) *iff* $\mathbf{P}$ *is μ-directed closed, and* $(\mathbf{P},\mathscr{D})$ *satisfies* (I7) *iff* $\mathbf{P}$ *is strongly μ-directed closed.*

PROOF. (a) The proof of the left-to-right implication, read backwards, proves the right-to-left implication, so we assume $(\mathbf{P},\mathscr{D})$ satisfies (I1), (I3)–(I6) and (I8)–(I10). Let $\mathscr{T}=\{p\in P^*:\operatorname{rlm} p\in\mu\}$, and for $p\in\mathscr{T}$, let $\lg p=\operatorname{rlm}(p)$. If $s:\lg p\to\mu^+$ is o.p., let $p(s)=\sigma_s(p)$. If there's more than one $p\in\mathscr{T}$ with $\operatorname{rlm} p=0$, then throw away all but one; if there's none, add one. In either case call the one we now have $O_{\mathscr{T}}$. The verifications are clear, and are left to the reader.

(b) The proof of (b) is clear from the above and from (3.10) (1), (2).

(3.14.3) Velleman proves (Definition 1.1, Lemmas 1.2–1.5, Lemma 1.7 and Theorem 1.6 of Chapter II).

LEMMA. *Let* $(\mathbf{P}',\mathscr{D}')$ *be almost μ-indiscernible and let* $(E'_\zeta:\zeta<\mu)$ *be a sequence of sets dense in* $\mathbf{P}'$. *Then there's* $(\mathbf{P},\mathscr{D})$ *which is μ-indiscernible and satisfies* (I7)–(I10) (*as well as the additional properties* (I11), (I12)) *with the property: If there's G which is $\mathbf{P}$-generic$/\mathscr{D}$ then there's an ideal G' in $\mathbf{P}'$ which* *meets all the D'_α. Further, if G is μ-complete, then so is G', and in this case G' also meets the E'_ζ.*

In his proof, Velleman assumes (I2) holds for $(\mathbf{P}', \mathcal{D}')$ and concludes that the $(\mathbf{P}, \mathcal{D})$ he constructs also satisfies (I2) as well as (I7). However, examination of his proof shows that if we no longer require (I2), (I7) for $(\mathbf{P}, \mathcal{D})$ then only a rather weak closure property need be imposed on the $\mathbf{P}'_\alpha$ $(\alpha < \mu)$. Further, the $\mathbf{P}_\alpha$ $(\alpha < \mu)$ (and hence $\mathbf{P}$, in the presence of (I8)–(I10)) inherit μ-strategic closure from the $\mathbf{P}'_\alpha$. This will be important in the sequel. The sets $(E'_\zeta : \zeta < \mu)$ correspond in a clear fashion to uniform dense sets in our treatment.

Velleman's forcing principle is then:

(*) Whenever $(\mathbf{P}, \mathcal{D})$ are μ-indiscernible and also satisfy (I7)–(I12), there is a μ-complete ideal G which is $\mathbf{P}$-generic/$\mathcal{D}$.

By the above, this translates to:

(**) Whenever $\mathbf{P}$ is μ-special, Indiscernible, strongly μ-directed closed satisfying Extension and Amalgamation, there's μ-complete G which meets all the D_α.

§4. More examples

Example 1. *Forcing a weak $\square_\kappa$-principle with κ-directed closed conditions of size $<\kappa$*

(4.1) Let $\kappa > \omega$, κ regular. Consider the following weakened version of Jensen's principle $\square_\kappa$:

weak $\square_\kappa$: there is a subset $A \subseteq \kappa^+$ all of whose members are limit ordinals and which contains all $\alpha < \kappa^+$ with $\operatorname{cf}\alpha = \kappa$, and a sequence $(C_\alpha : \alpha \in A)$ such that for $\alpha \in A$,

(i) $C_\alpha \subseteq \alpha$ is club of order-type $\leqq \kappa$,

(ii) if β is a limit point of C_α then $\beta \in A$ and $C_\beta = C_\alpha \cap \beta$.

If we require that A consist of *all* the limit ordinals $< \kappa^+$, then we get back $\square_\kappa$. For $\kappa = \aleph_1$, weak $\square_{\aleph_1} \Rightarrow \square_{\aleph_1}$ since off A we can fill in by supplying cofinal subsets of order-type ω. It is easy to derive weak $\square_\kappa$ directly from a $(\kappa, 1)$-morass, so the following example, forcing weak $\square_\kappa$ with conditions of size $<\kappa$, is presented essentially for motivation, but will also figure in §7.

(4.2) Definition. $p \in \bar{P}$ iff $p = (a, c)$, where $a \in [\kappa^+]^{<\kappa}$, $c = (c_\alpha : \alpha \in \operatorname{Lim} \cap a)$ are such that: (i) $a \neq \varnothing \Rightarrow$ there is $\alpha \in \operatorname{Lim} \cap a$ with $\operatorname{cf}\alpha = \kappa$,

(ii) for $\alpha \in \mathrm{Lim} \cap a : c_\alpha$ is a closed subset of α which is cofinal in α if $\mathrm{cf}\,\alpha < \kappa$, and such that if $\beta \in c_\alpha$ then $\beta \in a$, and if $\beta \in c^*_\alpha$ then $c_\beta = c_\alpha \cap \beta$; further (iii) if $\mathrm{cf}\,\alpha = \kappa$, then $\max c_\alpha = \sup a \cap \alpha$.

$(a,c) \leqq (a',c')$ iff $a \subseteq a'$ and for $\alpha \in a$, c'_α is an end-extension of c_α.

(4.3) PROPOSITION. *$\bar{\mathbf{P}}$ is κ-closed with least upper bounds.*

PROOF. If $((a^i,c^i): i<\gamma)$ is an increasing chain with $\gamma<\kappa$, then set $\bar{a} = \bigcup_{i<\gamma} a^i$, and if $\alpha \in \mathrm{Lim} \cap a^i$, set $\bar{c}_\alpha = \bigcup_{i \leqq j < \gamma} c^i_\alpha$. If $\alpha \in \mathrm{Lim} \cap \bar{a}$ and $\mathrm{cf}\,\alpha = \kappa$, set $\delta_\alpha = \sup \bar{c}_\alpha$; set

$$a = \bar{a} \cup \{\delta_\alpha : \alpha \in \bar{a} \text{ and } \mathrm{cf}\,\alpha = \kappa\},$$

and for $\alpha \in \mathrm{Lim} \cap \bar{a}$ and $\mathrm{cf}\,\alpha = \kappa$, set $c_{\delta_\alpha} = \bar{c}_\alpha$, $c_\alpha = c_\alpha \cup \{\delta_\alpha\}$. Clearly (a,c) is the lub of the (a^i,c^i).

(4.4) PROPOSITION (Amalgamation Property). *If (a,c), $(a',c') \in \bar{P}$, if a, a' have the strong Δ-property and c, c' agree on* $\mathrm{Lim} \cap a'$ *then (a,c), (a',c') are compatible.*

PROOF. $(a \cup a', c \cup c')$ is a condition, unless, letting $(a \setminus a') \ll (a' \setminus a)$ and letting $\alpha' = \inf(a' \setminus a)$, $\alpha' \in \mathrm{Lim}$ and $\mathrm{cf}\,\alpha' = \kappa$. In this case, simply add $\max a$ to $c'_{\alpha'}$.

(4.5) PROPOSITION (Extension Property). *If $(a,c) \in \bar{P}$, $\beta < \alpha < \kappa^+$ and $\mathrm{cf}\,\alpha = \kappa$ there is $(a',c') \geqq (a,c)$ with $\alpha \in a'$ and* $\max c'_\alpha \geqq \beta$.

PROOF. Clear.

(4.6) DEFINITION. Let X' be the set of ordinals less than κ^+ which are (ordinal) multiples of κ and let X be the set of κ-cofinal limit ordinals less than κ (so $X \subseteq X'$). $a \in [\kappa^+]^{<\kappa}$ is *nice* iff $\alpha \in (a \cap X') \setminus X \Rightarrow \alpha \in (a \cap X)^*$, and $a = \bigcup_{\alpha \in a \cap X'} (a \cap [\alpha, \alpha+\kappa))$. $P = \{(a,c) \in \bar{P} : a \text{ is nice}\}$. $\mathbf{P} = (P, \geqq)$.

(4.7) It's easy to see that (4.3)–(4.5) go through with $\mathbf{P}$ replacing $\bar{\mathbf{P}}$. To finish this example, we shall content ourselves with briefly describing the set of terms, how terms act on sequences to give conditions, and which densities are covering systems. The indiscernibility property will be clear.

Let $(a,c) \in P$. Let $s = (\alpha_i : i < \mathrm{o.t.}(a \cap X))$ increasingly enumerate $a \cap X$. For $i < \mathrm{o.t.}(a \cap X)$, let $\sigma_s^{-1}(\alpha_i) =$ the ith element of X, $\sigma_s^{-1}(0) = 0$, and if $\beta \in (a \cap X^*) \setminus X$ (so, since a is nice, $\beta \in (a \cap X)^*$), $\sigma_s^{-1}(\beta) = \sup_{\alpha \in a \cap X \cap \beta} \sigma_s^{-1}(\alpha)$. Finally if $\gamma < \kappa$, $\alpha \in a \cap X'$, $\sigma_s^{-1}(\alpha+\gamma) = \sigma_s^{-1}(\alpha) + \gamma$. Then $\bar{a} = \sigma_s^{-1}[a]$ and for $\alpha \in a$, $\bar{c}_{\sigma_s^{-1}(\alpha)} = \sigma_s^{-1}[c_\alpha]$.

$(\bar{a}, \bar{c})$ is the contraction of (a,c) and $\mathcal{T}$, the set of terms, is just the set of contractions which is the same as $\{(a,c)\in P : a \cap X$ is an initial segment of $X\}$. The action of terms on increasing sequences is obtained by reversing the preceding construction.

If $i<\kappa$, $\tau=(\bar{a},\bar{c})\in\mathcal{T}$, set $\tau\in D_i$ if ($\alpha\in\bar{a}$ and $\operatorname{cf}\alpha=\kappa$) $\Rightarrow$ $\bar{c}_\alpha$ has order-type $\geqq i$. Clearly each D_i is dense in $\mathcal{T}$ and if G is an ideal meeting each D_i^* uniformly, if $A=\bigcup_{p\in G}a^p$, $C_\alpha=\bigcup\{c^p_\alpha : \alpha\in a^p, p\in G\}$ for $\alpha\in A$, then A contains all points of $\operatorname{cf}\kappa$ and (i) of (4.1) holds. It remains only to see that if $\alpha\in A$ and $\operatorname{cf}\alpha=\kappa$, then C_α is cofinal in α. Let $\beta=\sup C_\alpha$. Clearly C_α has order-type κ so that $\operatorname{cf}\beta=\kappa$. But then if $\beta<\alpha$, there is $(a,c)\in G$ with $\beta,\alpha\in a$, and so $\max c_\alpha\geqq\beta$ which is impossible.

EXAMPLE 2. *Two-gap–two-cardinal models via forcing, à la Burgess*

(4.8) The following is, again, essentially for motivation, as it furnishes a new proof of Jensen's two-gap–two-cardinal theorem from morasses, but no new result. Accordingly, we permit ourselves a certain latitude in passing over some details which the interested reader can verify easily enough. The forcing conditions presented below are due to Shelah. Burgess [2] first exploited similar conditions to add two-gap–two-cardinal models directly via forcing. D. Velleman has found a similar modification of Burgess's conditions. It would be helpful if the reader were familiar with Burgess's paper [2] or with the model theoretic lemmas involved in Jensen's proof of the two-gap theorem from morasses (see [3]).

Let κ be an infinite cardinal, and assume GCH. Let T be a first-order theory, in countable language $\mathcal{L}$ which has a distinguished unary predicate symbol $\boldsymbol{U}_0$, such that for some infinite λ, T has a (λ^{++},λ) model. Add to $\mathcal{L}$ another unary predicate symbol $\boldsymbol{U}_1$ and two binary predicate symbols, $\lhd$, $\boldsymbol{E}$, and let T' be the complete theory $\mathrm{Th}(M)$, where $M=(H_{\lambda^{++}},\lambda,\lambda^+,\lhd,\in,\cdots\}$, where λ interpretes $\boldsymbol{U}_0$, λ^+ interpretes $\boldsymbol{U}_1$, $\lhd$ interpretes $\blacktriangleleft$, $\in$ interpretes $\boldsymbol{E}$, $\lhd$ is a well order of $H_{\lambda^{++}}$ in type λ^{++} which extends $\in$ and has λ^+ as an initial segment, and where some (λ^{++},λ) model of T has been isomorphed onto $(H_{\lambda^{++}},\lambda,\cdots)$. Thus $T'\supseteq T$.

Suppose $\mathfrak{A}\models T'$. Let $\boldsymbol{U}_2^{\mathfrak{A}}=|\mathfrak{A}|\setminus\boldsymbol{U}_1^{\mathfrak{A}}$, and for $b\in|\mathfrak{A}|$, let $Y_b=\{a\in|\mathfrak{A}| : a\blacktriangleleft^{\mathfrak{A}}b\}$. Then, there is a unique $c\in|\mathfrak{A}|$ such that for all $b\in|\mathfrak{A}|$, $b\boldsymbol{E}^{\mathfrak{A}}c\Leftrightarrow b\in Y_a$. Further, for this c there is unique $f\in|\mathfrak{A}|$ which is minimal in the sense of $\blacktriangleleft^{\mathfrak{A}}$ for the property: $\mathfrak{A}\models$"$f: c\xrightarrow{1\text{-}1}\boldsymbol{U}_1$". This f gives rise in the obvious way to a 1-1 map of Y_a into $\boldsymbol{U}_1^{\mathfrak{A}}$ which is $\mathfrak{A}$-definable from a. Accordingly:

(a) if $\mathfrak{A}<\mathfrak{B}$, $\mathfrak{A}\models T'$ and $\boldsymbol{U}_1^{\mathfrak{A}}=\boldsymbol{U}_1^{\mathfrak{B}}$, then $\mathfrak{B}$ is an end-extension of $\mathfrak{A}$ in the sense of $\blacktriangleleft^{\mathfrak{B}}$ (and hence in the sense of $\boldsymbol{E}^{\mathfrak{B}}$).

(4.9) Let M, T' be as in (4.8). Since $M \models$ "Reflection Scheme for $\mathcal{L}'$-formulas", we have for all $\mathfrak{A} \models T'$, all $\mathcal{L}'$-formulas $\varphi(v)$:

(a) $\mathfrak{A} \models (\forall a)(\exists b)(a \lhd b \wedge (\forall v \lhd b)(\varphi(v) \Leftrightarrow \varphi^{Y_b}(v)))$, where φ^{Y_b} is the relativization of φ to Y_b.

Therefore, if $\mathfrak{A}$ is saturated, if $a \in |\mathfrak{A}|$, then $ES_a(w) = \{(\forall v \lhd w)(\varphi(v) \Leftrightarrow \varphi^{Y_w}(v)) : \varphi(v) \text{ an } \mathcal{L}'\text{-formula}\} \cup \{a \lhd w\}$ is realized, i.e. $\{b \in |\mathfrak{A}| : \mathfrak{A} \mid Y_b < \mathfrak{A}\}$ is cofinal in $(|\mathfrak{A}|, \lhd^{\mathfrak{A}})$. Also, by saturation, $\kappa = \operatorname{card}|\mathfrak{A}|$ is regular and $\kappa = \operatorname{cf}(|\mathfrak{A}|, \lhd^{\mathfrak{A}})$.

Now suppose that $\mathfrak{A} \models ES(b)$ and that $\Gamma(w)$ is a 1-type over $\mathfrak{A}$ with $< \kappa$ constants, all from Y_b. Then $\Gamma(w) \cup \{w \lhd b\}$ is also a 1-type over $\mathfrak{A}$, as is easily seen applying (a) to $(\exists w)(\theta_0(w) \wedge \cdots \wedge \theta_{n-1}(w))$, where $\theta_0, \cdots, \theta_{n-1} \in \Gamma(w)$. Hence $\Gamma(w) \cup \{w \lhd b\}$ is realized in $\mathfrak{A}$ and hence in $\mathfrak{A} \mid Y_b$: i.e. $\mathfrak{A} \mid Y_b$ is saturated.

If $\mathfrak{A} \models T'$, $\operatorname{card}|\mathfrak{A}| = \kappa$, then $\mathfrak{A}$ is *layered* iff $U_0^{\mathfrak{A}} = \kappa$, $U_1^{\mathfrak{A}}$ is an initial segment of κ^+, $U_2^{\mathfrak{A}} \subseteq \{\kappa^{++} \setminus \kappa^+\} \times \kappa^+$. For layered $\mathfrak{A}$, let $X_{\mathfrak{A}} = \{\alpha < \kappa^{++} : (\exists \beta < \kappa^+)(\alpha, \beta) \in U_2^{\mathfrak{A}})\}$, and let $\bar{X}_{\mathfrak{A}} = \{\alpha \in X_{\mathfrak{A}} : \alpha \text{ is a successor or cf } \alpha = \kappa^+\}$.

$\mathfrak{A}$ is nicely layered iff $\mathfrak{A}$ is layered, $\kappa^+ \in X_{\mathfrak{A}}$ and

(i) $\alpha \in X_{\mathfrak{A}} \Rightarrow (\alpha, 0) \in |\mathfrak{A}|$, and if $\alpha > \kappa^+$, then $Y_{(\alpha,0)} = U_1^{\mathfrak{A}} \cup \{(\alpha', \beta) \in |\mathfrak{A}| : \alpha' < \alpha\}$, and $\mathfrak{A} \models ES((\alpha, 0))$; further, $\alpha \in X_{\mathfrak{A}} \setminus \bar{X}_{\mathfrak{A}} \Rightarrow X_{\mathfrak{A}}$ is cofinal in α.

(ii) $X_{\mathfrak{A}} = \bar{X}_{\mathfrak{A}} \cup (\bar{X}_{\mathfrak{A}})^*$, so that, in particular, $X_{\mathfrak{A}}$ is closed.

Note that by (i), if (α', β'), $(\alpha, \beta) \in |\mathfrak{A}|$ and $\alpha' < \alpha$, then $(\alpha', \beta') \lhd^{\mathfrak{A}} (\alpha, \beta)$.

(4.10) Now assume $\kappa > \omega$ is regular. Fix $\mathfrak{A}_0 \models T'$ of cardinal κ, $U_0^{\mathfrak{A}_0} = \kappa$, $U_1^{\mathfrak{A}_0}$ an initial segment of κ^+, $|\mathfrak{A}_0| \subseteq \kappa^+$.

Definition. Set $\mathfrak{A} \in P$ iff $\mathfrak{A} = \mathfrak{A}_0$, or $\mathfrak{A} \models T'$ is saturated and nicely layered. For $\mathfrak{A}$, $\mathfrak{B} \in P$, $\mathfrak{A} \leqq \mathfrak{B}$ iff $\mathfrak{A} = \mathfrak{A}_0$ or $\mathfrak{A} < \mathfrak{B}$. $\mathbf{P} = (P, \geqq)$. Let $P' = P \setminus \{\mathfrak{A}_0\}$.

(4.11) Jensen's crucial model-theoretic lemma is (cf. [3], chapter 14, lemma 7, pp. 179–181, or [2], lemmas 1.2, 2.2, pp. 3, 4):

Lemma. *Suppose $\mathfrak{A} \models T'$, $\mathfrak{A}$ is U_0-saturated,* $\operatorname{card}|\mathfrak{A}| = \kappa$, $a \in U_2^{\mathfrak{A}}$ *and* $\mathfrak{A} \models ES(a)$. *Then there are $\mathfrak{A}'$, j such that $\mathfrak{A}'$ is U_0-saturated,* $\operatorname{card}|\mathfrak{A}'| = \kappa$, $U_0^{\mathfrak{A}} = U_0^{\mathfrak{A}'}$ *and*:

(i) *$j : \mathfrak{A} \to \mathfrak{A}'$ is an elementary embedding,* $j \mid Y_a = \operatorname{id} \mid Y_a$,

(ii) *$\mathfrak{A} < \mathfrak{A}' \mid Y_{j(a)}$ (so that in particular $x \in |\mathfrak{A}| \Rightarrow x \lhd^{\mathfrak{A}'} j(a)$).*

Note that, since $\mathfrak{A} \models ES(a)$ and since j is elementary, $\mathfrak{A}' \models ES(j(a))$, so that $\mathfrak{A}' \mid Y_{j(a)} < \mathfrak{A}'$, whence $\mathfrak{A} < \mathfrak{A}'$, by (ii).

(4.12) We shall need the following consequence of (4.11).

COROLLARY. *Suppose* $\mathfrak{B}$ *is layered and that* (i) *in the definition of nicely layered holds for* $\mathfrak{B}$. *Suppose* $\mathfrak{B} < \mathfrak{A}$, $\mathfrak{A}$ *is* U_0*-saturated,* $\operatorname{card}|\mathfrak{A}| = \kappa$, $U_0^{\mathfrak{B}} = U_0^{\mathfrak{A}} = \kappa$. *Suppose* $\alpha \notin X_{\mathfrak{B}}$, $\kappa^+ < \alpha$ *and* α *has the following properties*:

(i) $(\alpha \in \mathrm{Lim} \wedge \operatorname{cf}\alpha < \kappa) \Rightarrow (X_{\mathfrak{B}} \cap \alpha)$ *is cofinal in* α.

(ii) $(\alpha$ *a successor ordinal or* $\operatorname{cf}\alpha = \kappa) \Rightarrow \mathrm{o.t.}(X_{\mathfrak{B}} \cap \alpha) \notin \mathrm{Lim}$.

Then there's layered, saturated $\mathfrak{B}'$ *with* $\operatorname{card}|\mathfrak{B}'| = \kappa$, $U_0^{\mathfrak{B}'} = \kappa$, $\mathfrak{B} < \mathfrak{B}'$ *and such that* (i) *of the definition of nicely-layered holds for* $\mathfrak{B}'$. *Further, there is* $h : \mathfrak{A} \to \mathfrak{B}'$ *an elementary embedding with* $h \restriction |\mathfrak{B}| = \mathrm{id} \restriction |\mathfrak{B}|$. *Finally, let* $\bar{\alpha} = \sup X_{\mathfrak{B}}$. *Then* $\mathfrak{B}'$ *can be taken with* $X_{\mathfrak{B}} \cup \{\alpha\} \subseteq X_{\mathfrak{B}'} \subseteq X_{\mathfrak{B}} \cup \{\alpha, \bar{\alpha}\}$.

PROOF. Note that by our assumptions on α, if $\alpha > \bar{\alpha}$, then (α is a successor or $\operatorname{cf}\alpha = \kappa$) and $\bar{\alpha} \in X_{\mathfrak{B}}$. Let $\beta^* = \inf(X_{\mathfrak{B}} \setminus \alpha)$ if $\alpha < \bar{\alpha}$; otherwise, let $\beta^* = \inf(X_{\mathfrak{B}} \setminus \kappa^+ + 1)$; set $a = (\beta^*, 0)$.

Note that by (i) in the definition of nicely-layered, $\beta^* \in \bar{X}_{\mathfrak{B}}$.

We'll apply (4.11) to $\mathfrak{A}$ and a to obtain $\mathfrak{A}'$, j as in (4.11). By U_0-saturation of $\mathfrak{A}'$, find $\mathfrak{A}''$, a saturated elementary extension of $\mathfrak{A}'$ of power κ having the same U_0. We'll define $h' : \mathfrak{A}'' \to \mathfrak{B}'$ an isomorphism. We'll define h' by describing how to rename elements of $|\mathfrak{A}''|$ in a 1-1 fashion. When $\alpha \geqq \bar{\alpha}$ we'll have $h' \restriction |\mathfrak{B}| = \mathrm{id} \restriction |\mathfrak{B}|$, and we'll take $h = h'$. When $\alpha < \bar{\alpha}$, we'll have $h' \circ j \restriction |\mathfrak{B}| = \mathrm{id} \restriction |\mathfrak{B}|$, and we'll take $h = h' \circ j$. So, define k to be $\mathrm{id} \restriction |\mathfrak{B}|$ if $\alpha \geqq \bar{\alpha}$, and $k = j \restriction |\mathfrak{B}|$ otherwise.

So, h' is the identity on $U_1^{\mathfrak{B}}$, and we use the least available members of κ^+ to rename the elements of $U_1^{\mathfrak{A}''} \setminus U_1^{\mathfrak{B}}$. We shall first say how to proceed when $\alpha \geqq \bar{\alpha}$. Then, for the case when $\alpha < \bar{\alpha}$, we shall assume without loss of generality that $\bar{\alpha} \in X_{\mathfrak{B}}$, since by what we'll have already proved, we could simply add $\bar{\alpha}$ first, and then proceed to add α. Before this division into cases, we make some definitions which hold in all cases. Let $x \in U_2^{\mathfrak{A}''}$. There are two possibilities:

(a) there are $\beta \in X_{\mathfrak{B}}$, γ, $\gamma' < \kappa^+$, such that $(\beta, \gamma) \lhd^{\mathfrak{B}} (\beta, \gamma')$, and x is a member of the half-open $\lhd^{\mathfrak{A}''}$-interval $[k(\beta, \gamma), k(\beta, \gamma'))$; let $\beta(x) =$ the unique such β;

(b) otherwise.

Our treatment of (b) will differ, according to the cases, but if x falls under (a), we shall always rename x to be $(\beta(x), \xi)$ for some $\xi \in \kappa^+$ with $(\beta(x), \xi) \notin |\mathfrak{B}|$. The renaming will choose distinct ξ's for distinct x's.

So, first suppose that $\alpha = \bar{\alpha}$ (so $k = \mathrm{id} \restriction |\mathfrak{B}|$). If $x \in U_2^{\mathfrak{A}''}$ and falls under (b) above, then for all $y \in |\mathfrak{B}|$, $y \lhd^{\mathfrak{A}''} x$. Rename all such x's to be (α, ξ) for some $\xi \in \kappa^+$; the renaming will choose distinct ξ's for distinct x's.

Suppose now that $\alpha > \bar{\alpha}$ (by hypothesis, in this case, $\bar{\alpha} \in X_{\mathfrak{B}}$, and again

$k = \mathrm{id} \restriction |\mathfrak{B}|$). If $x \in U_2^{\mathfrak{A}''}$ and falls under (b), we distinguish two further possibilities:

(bi) $x \lhd^{\mathfrak{A}''} j(a)$; then rename x to be $(\bar{\alpha}, \xi)$ for some $\xi \in \kappa^+$ such that $(\bar{\alpha}, \xi) \not\in |\mathfrak{B}|$; the renaming will choose distinct ξ's for distinct x's,

(bii) $j(a) = x$ or $j(a) \lhd^{\mathfrak{A}''} x$; then rename x to be (α, ξ) for some $\xi < \kappa^+$, and require that $j(a)$ is renamed to be $(\alpha, 0)$, as well as that distinct ξ's be chosen for distinct x's.

Now suppose that $\alpha < \bar{\alpha}$, and recall that we're assuming $\bar{\alpha} \in X_{\mathfrak{B}}$, and that $k = j \restriction |\mathfrak{B}|$. The proof now divides, depending on whether $\alpha = \sup(X_{\mathfrak{B}} \cap \alpha)$. If so, and if $x \in U_2^{\mathfrak{A}''}$ and falls under (b), we distinguish the following possibilities:

(bi′) for all $\beta \in X_{\mathfrak{B}} \cap \alpha$, $(\beta, 0) \lhd^{\mathfrak{A}''} x$, but $x \lhd^{\mathfrak{A}''} j(a)$; rename x to be (α, ξ) for some ξ, etc., ...,

(bii′) $j(\bar{\alpha}, 0) \lhd^{\mathfrak{A}''} x$; rename x to be $(\bar{\alpha}, \xi)$ for some $\xi < \kappa^+$; the ξ's are required to be distinct, and also be distinct from those ξ's chosen for y's falling under (a) with $\beta(y) = \bar{\alpha}$.

Finally, if $\alpha < \bar{\alpha}$ and $\alpha > \sup(X_{\mathfrak{B}} \cap \alpha)$, we distinguish the possibilities:

(bi″) $x \lhd^{\mathfrak{A}''} j(a)$; then rename x to be (α, ξ), etc.;

(bii″) $j(\bar{\alpha}, 0) \lhd^{\mathfrak{A}''} x$; rename x to be $(\bar{\alpha}, \xi)$, with the stipulations of (bii′).

This completes the definition of h' in all cases. The verifications are left to the reader.

(4.13) As a result we have:

Lemma. **P** *is* κ^+*-directed closed.*

Proof. The union of a $<$-directed system of size $\leqq \kappa$ of U_0-saturated models of T' having the same interpretation of U_0 is U_0-saturated by Chang's trick, which works in this context just as well as for chains. Call this union $\mathfrak{B}$. We know that $X_{\mathfrak{B}} \setminus \bar{X}_{\mathfrak{B}} \subseteq (\bar{X}_{\mathfrak{B}})^*$. Let $(\alpha_i : i < \theta)$ enumerate $(\bar{X}_{\mathfrak{B}})^* \setminus X_{\mathfrak{B}}$, with $\alpha_0 = \sup X_{\mathfrak{B}}$ if $\sup X_{\mathfrak{B}} \not\in X_{\mathfrak{B}}$. Set $\mathfrak{B}_0 = \mathfrak{B}$. Using (4.12) at successors, and taking unions at limits, we build an elementary tower of U_0-saturated models of power κ, $(\mathfrak{B}_i : i \leqq \theta)$ s.t. $\alpha_i \in X_{\mathfrak{B}_{i+1}}$, and $U_0^{\mathfrak{B}_i} = \kappa$. We let $\mathfrak{A} = \mathfrak{B}_\theta$; thus $\mathfrak{A}$ is U_0-saturated, and clearly $\mathfrak{A}$ is layered, (i) in the definition of nicely-layered holds. By construction $X_{\mathfrak{A}} = X_{\mathfrak{B}} \cup (X_{\mathfrak{B}})^*$ and $\bar{X}_{\mathfrak{A}} = \bar{X}_{\mathfrak{B}}$; so in fact $\mathfrak{A}$ is nicely layered. It will then suffice to find saturated $\mathfrak{A}'$ of power κ with $U_0^{\mathfrak{A}'} = \kappa$, $\mathfrak{A} < \mathfrak{A}'$ and $\mathfrak{A}'$ nicely layered with $X_{\mathfrak{A}'} = X_{\mathfrak{A}}$. Clearly we can find $\mathfrak{A}''$ with all of these properties except being nicely layered and having $X_{\mathfrak{A}''} = X_{\mathfrak{A}}$. However, having such an $\mathfrak{A}''$ we can obtain $\mathfrak{A}' \sim \mathfrak{A}''$ which is nicely layered with $X_{\mathfrak{A}'} = X$ by renaming elements of $\mathfrak{A}''$ along the lines of (4.12). The details are left to the reader.

(4.14) Now let's see that $\mathbf{P}$ is κ^+-special. Let $\bar{X} = \{\alpha < \kappa^{++} \setminus \kappa^+ : \alpha$ is successor or $\operatorname{cf} \alpha = \kappa^+\}$ and let $(x_i : i < \kappa^{++})$ be the increasing enumeration of $\bar{X}$. If $\mathfrak{A} \in P'$, let $(y_i : i < \theta_{\mathfrak{A}})$ be the increasing enumeration of $\bar{X}_{\mathfrak{A}}$ and let $s_{\mathfrak{A}} : \theta_{\mathfrak{A}} \to \kappa^{++}$ be such that $y_i = x_{s_{\mathfrak{A}}(i)}$ for $i < \theta_{\mathfrak{A}}$. Let $\pi'_{\mathfrak{A}}$, $\pi_{\mathfrak{A}}$, $\bar{\mathfrak{A}}$ be such that $\pi_{\mathfrak{A}} : \mathfrak{A} \leftrightarrow \bar{\mathfrak{A}}$, $\pi_{\mathfrak{A}} \mid U_1^{\mathfrak{A}} = \mathrm{id} \mid U_1^{\mathfrak{A}}$, for $i < \theta_{\mathfrak{A}}$, $\pi'_{\mathfrak{A}}(y_i) = x_i$, for $y \in (\bar{X}_{\mathfrak{A}})^*$, $\pi'_{\mathfrak{A}}(y) = \sup_{y' \in \bar{X}_{\mathfrak{A}} \cap y} \pi'(y')$, and for $(\alpha, \beta) \in U_2^{\mathfrak{A}}$, $\pi_{\mathfrak{A}}(\alpha, \beta) = (\pi'_{\mathfrak{A}}(\alpha), \beta)$. Then $\mathcal{T} = \{\bar{\mathfrak{A}} : \mathfrak{A} \in P'\} \cup \{\mathfrak{A}_0\}$, for $\mathfrak{A} \in P'$, $\lg \bar{\mathfrak{A}} = \theta_{\mathfrak{A}}$. Also, $\lg \mathfrak{A}_0 = 0$. For $\mathfrak{A} \in P'$, for $s \in [\kappa^{++}]^{\lg \mathfrak{A}}$, obtain $\bar{\mathfrak{A}}(s)$ by reversing the above construction and let $\mathfrak{A}_0(\varnothing) = \mathfrak{A}_0$. The indiscernibility property is clear.

(4.15) LEMMA. $\mathbf{P} \in \mathcal{S}_{\kappa^+}$.

PROOF. First note that $\mathbf{P}$ is strongly κ-directed closed, since the upper bound $\mathfrak{A}'$ constructed in (4.13) has the property that $\bar{X}_{\mathfrak{A}'} = \bigcup_{\mathfrak{B} \in D} \bar{X}_{\mathfrak{B}}$. It remains to prove the Extension and Amalgamation Properties. These follow readily from (4.11) and the arguments of (4.12), or see [3], lemmas 6, 7, pp. 179–180, or [2], lemmas 2.5, 2.7.

For $\kappa = \omega$, use recursively-saturated and recursively-U_0-saturated models of T' as in [2].

(4.16) The sufficiently generic sets here are particularly simple. Any ideal meeting all the D_α is sufficiently generic. Hence, Theorem 1(b) and GCH yield a new proof of Jensen's 2-Gap–2-Cardinal Theorem. This is because any such G is a directed (for $<$) system of models of T' all having the same interpretation of U_0. Hence $\bigcup G \models T'$, $U_0^{\cup G} = \kappa$, and for all $s \in [\kappa^{++}]^{<\kappa^+}$ there is $s' \in [\kappa^{++}]^{<\kappa^+}$ with range $s \subseteq$ range s' and $\mathfrak{A} \in G$ with $\{x_{s'_i} : i < \lg s'\} = \bar{X}_{\mathfrak{A}}$. But then clearly $\operatorname{card}(|\bigcup G|) = \kappa^{++}$, and so the $\mathcal{L}$-reduct of $\bigcup G$ is a (κ^{++}, κ)-model of T.

§5. Adding morasses by S-forcing

In this section we prove one direction of the main theorem by exhibiting, for each regular uncountable μ, a $\mathbf{P} \in \mathcal{S}_\mu$ which adds a $(\mu, 1)$-morass; in fact with the property that there are μ uniform dense subsets of $\mathbf{P}$ which, if met uniformly, guarantee the existence of a $(\mu, 1)$-morass. Thus, S_μ guarantees the existence of $(\mu, 1)$-morasses. Fix μ regular and uncountable.

(5.1) Our strategy for adding a $(\mu, 1)$-morass is to add a simpler object which we then "thin" to obtain a $(\mu, 1)$-morass. This approach is, essentially, due to Jensen and is basically that of [12]. The novelty here is that D. Velleman found a far simpler set of conditions which he has kindly permitted us to present here and in Stanley's articles [12], [13]. This material will also appear in [14].

The basic difference is that rather than starting from a tree with maps along it which is provided by a $\square_\mu$-sequence, and then picking out a subtree (the Jensen method, as in [11]), Velleman's conditions leave us entirely free to assemble the tree and the maps. As a result, the conditions and the generic object have a more purely "combinatorial" flavor, but more importantly, the conditions are μ-directed closed, whereas the Jensen conditions, involving as they do first forcing $\square_\mu$, are not even μ-closed, unless $\mu = \aleph_1$.

The same model-theoretic techniques as in [12] are used for the thinning, for which see [12]. We don't know whether a similar simplification is to be had for forcing higher-gap morasses.

(5.2) DEFINITION. A $(\mu, 1)$-*premorass* is a triple $(T, \dashv, f_{xy})_{x \dashv y}$ such that

(i) $(T, \dashv)$ is a tree; $T \subseteq (\mu+1) \times \mu^+$,

$$\{\mu\} \times \mu^+ \subseteq T, \quad (\operatorname{dom} T) \cap \mu = \mu;$$

(ii) for $\alpha \in (\operatorname{dom} T) \cap \mu$, $0 < \gamma_\alpha = \{\tau : (\alpha, \tau) \in T\} \in \mu$; for $x \in T$, set $x = (\mathrm{l}(x), \mathrm{o}(x))$ (*warning*: $\mathrm{l}(x)$ *need not coincide with the level of* x *in* $(T, \dashv)$); if $x \dashv y$ then $\mathrm{l}(x) < \mathrm{l}(y)$, $\mathrm{o}(x)$, $\mathrm{o}(y)$ are the *same kind* of ordinal, and $f_{xy} : \mathrm{o}(x) \dashv \mathrm{o}(y)$ is a nice map; further, if $\mathrm{o}(x) = \alpha + 1$, $\mathrm{o}(y) = \beta + 1$, then $f_{xy}(\alpha) = \beta$ (thus we may conventionally set $f_{xy}(\mathrm{o}(x)) = \mathrm{o}(y)$ and niceness is preserved); $(f_{xy} : x \dashv y)$ is commutative;

(iii) if $x \dashv y$, $\eta < \mathrm{o}(x)$, $w = (\mathrm{l}(x), \eta)$, $z = (\mathrm{l}(y), f_{xy}(\eta))$, then $w \dashv z$ and $f_{wz} = f_{xy} \restriction \eta$;

(iv′) $\{\mathrm{l}(x) : x \dashv y\}$ is either empty or a coinitial segment of $(\operatorname{dom} T) \cap \mathrm{l}(y)$, and is non-empty if $\mathrm{l}(y) \in \mathrm{Lim}$ and there is $\eta > \mathrm{o}(y)$ s.t. $(\mathrm{l}(y), \eta) \in T$;

(v′) if $\mathrm{l}(y) \in \mathrm{Lim}$ and y is non-minimal in $\dashv$,

$$\mathrm{o}(y) = \bigcup_{x \dashv y} f_{xy}{}''\mathrm{o}(x);$$

(vi) if $\mathrm{o}(x) \in \mathrm{Lim}$, $x \dashv y$ and $\lambda = \sup f_{xy}{}''\mathrm{o}(x) < \mathrm{o}(y)$ then, setting $z = (\mathrm{l}(y), \lambda)$, $x \dashv z$, $f_{xz} = f_{xy}$.

(5.3) LEMMA (Jensen). *If there is a* $(\mu, 1)$-*premorass, there is a* $(\mu, 1)$-*morass.*

PROOF. By thinning. See [12].

(5.4) DEFINITION. $p = (t, \dashv, f_{xy})_{x \dashv y} \in P$ iff

(i): $t \subseteq (\mu+1) \times \mu^+$, $\operatorname{card} t < \mu$, $t \neq \varnothing \Rightarrow (\mu, 0) \in t$ and $(\operatorname{dom} t) \cap \mu$ is a successor ordinal, $(t, \dashv)$ is a tree;

(ii): (ii), (iii), (iv′), (vi) of (5.2) hold, replacing T by t (note that if $\mathrm{l}(y) = \mu$, in (iii) this means that $(\mu, f_{xy}(\eta)) \in t$, and in (vi) this means that $(\mu, \lambda) \in t$);

(iii): (v′) of (5.2) holds, with the additional restriction that $l(y) < \mu$.

If $p \in P$, set $p = (t^p, \dashv^p, f^p_{xy})_{x \dashv^p y}$, and define S^p_μ, and for $\alpha \in (\operatorname{dom} t^p) \cap \mu$, γ^p_α, by setting $t^p = (\bigcup_{\alpha \in (\operatorname{dom} t^p) \cap \mu} \{\alpha\} \times \gamma^p_\alpha) \cup \{\mu\} \times S^p_\mu$. Also, for $\alpha \leqq \mu$, set $t^p \mid \alpha = \{x \in t^p : l(x) < \alpha\}$, and set

$$p \mid \alpha = (t^p \mid \alpha, \dashv^p \mid (t^p \mid \alpha), f^p_{xy})_{x \dashv^p y, l(y) < \alpha}.$$

Then, for $p, q \in P$, set $q \geqq p$ iff, setting $\alpha = \max((\operatorname{dom} t^p) \cap \mu) + 1$,

(iv) $p \mid \alpha = q \mid \alpha$, $S^p_\mu \subseteq S^q_\mu$ and for $x, y \in t^p$, if $l(y) = \mu$, then $x \dashv^p y$ iff $x \dashv^q y$ and if $x \dashv^p y$, $f^p_{xy} = f^q_{xy}$. Set $\mathbf{P} = (P, \geqq)$.

(5.5) Definition. Let $X = \{\eta \in \operatorname{Lim} \cap \mu^+ : \operatorname{cf} \eta = \mu\}$, and let $\tilde{X}$ be the set of multiples of μ which are less than μ^+. Let $S \in [\mu^+]^{<\mu}$. S is *acceptable* iff S is closed, for $\eta \in S \cap \tilde{X}$, $S \cap [\eta, \eta + \mu)$ is an initial segment of $[\eta, \eta + \mu)$ and $S = \bigcup\{S \cap [\eta, \eta + \mu) : \eta \in S \cap \tilde{X}\}$; if $\eta \in S \cap X$ let $\lambda_\eta = \sup S \cap \eta$.

The following proposition is now obvious.

Proposition. (a) *Whenever $A \in [\mu^+]^{<\mu}$, there is acceptable $S \supseteq A$.*

(b) *If $\mathcal{S}$ is a family of acceptable sets, directed under $\subseteq$ and* card $\mathcal{S} < \mu$, *set $\bar{S} = \bigcup \mathcal{S}$, and let $S = \bar{S} \cup \{\sup(\bar{S} \cap \eta) : \eta \in \bar{S} \cap X\} \cup \{\sup \bar{S}\}$. Then S is the smallest acceptable set $\supseteq \bar{S}$.*

(c) *If S is acceptable, let $S' = S \setminus \{\lambda_\eta : \eta \in S \cap X\}$. For $\eta \in S$, let $\delta_\eta =$* o.t.$(S' \cap \eta)$ *and let $f_\eta : \delta_\eta \to S' \cap \eta$ be the increasing enumeration; then f_η is nice, if $\eta \in \operatorname{Lim} \setminus X$ then* range f_η *is cofinal in η, if $f_\eta(\xi) = \eta'$ then $\xi = \delta_{\eta'}$, $f_{\eta'} = f_\eta \mid \xi$, and if $\eta \in X$, $S' \cap \lambda_\eta = S' \cap \eta$, $\delta_\eta = \delta_{\lambda_\eta}$, $f_\eta = f_{\lambda_\eta}$.*

(5.6) Definition. If $p \in P$, p is *acceptable* iff S^p_μ is acceptable, and, letting $S = S^p_\mu$, letting $\alpha = \max((\operatorname{dom} t^p) \cap \mu)$, and adopting the notation of (5.5), if $\eta \in S$, then $\gamma^p_\alpha = \{\delta_\eta : \eta \in S\}$ and, for $\eta \in S$, setting $x = (\alpha, \delta_\eta)$, $y = (\mu, \eta)$, then $x \dashv^p y$ and $f^p_{xy} = f_\eta$. Let $P' = \{p \in P : p \text{ is acceptable}\}$. $\mathbf{P}' = (P', \geqq)$.

(5.7) Lemma. *Suppose $p, q \in P'$ and $p \mid \mu = q \mid \mu$. Then S^p_μ, S^q_μ have the strong Δ-property iff p, q are compatible in* $\mathbf{P}'$.

Proof. If $p \mid \mu = q \mid \mu$ and S^q_μ, S^q_μ have the strong Δ-property then:

(*): whenever $\eta \in S^p_\mu \cap S^q_\mu$ and $x \in t^p \mid \mu$ then, setting $y = (\mu, \eta)$, $x \dashv^p y$ iff $x \dashv^q y$ and if $x \dashv^p y$ then $f^p_{xy} = f^q_{xy}$ (note further that (*) is necessary for p, q to be compatible).

So suppose S^p_μ, S^q_μ have the strong Δ-property. Suppose, e.g., that $S^p_\mu = a \cup b$, $S^q_\mu = a \cup c$, where $a \ll b \ll c$. By (ii), (iv), (v) of (5.4) it is easy to see that if η', η'' are the least elements of b, c, respectively, then $\eta', \eta'' \in X$. Let

$\sigma = \max S^p_\mu$. Let $S = S^p_\mu \cup \{\sigma + i : 1 \leqq i \leqq \omega\} \cup S^q_\mu$. Then, it's easy to see that S is acceptable and $\sigma + \omega = \lambda_{\eta''}$.

Set $t = t^p \restriction \mu \cup \{(\alpha+1, \delta_\eta) : \eta \in S\} \cup \{(\mu, \eta) : \eta \in S\}$, where $\alpha = \max((\operatorname{dom} t^p) \cap \mu)$, and for $\eta \in S$, δ_η is as in (5.5). For $x, y \in t$, set $x \dashv y$ iff:

(i) for $r \in \{p, q\}$, $x \dashv^r y$; then $f_{xy} = f^r_{xy}$, or

(ii) for $\eta \in S$, $y = (\mu, \eta)$, $x = (\alpha+1, \delta_\eta)$; then $f_{xy} = f_\eta$ (as in (5.5)), or

(iii) for $\eta \in S$, $r \in \{p, q\}$, $y = (\alpha+1, \delta_\eta)$, $x \dashv^r y' = (\mu, \eta)$; then $f_{xy} = f^{-1}_{yy'} \circ f_{xy'}$, or

(iv) $y = (\mu, \sigma+\omega)$, $x \dashv^q y' = (\mu, \eta'')$; then $f_{xy} = f_{xy'}$.

Clearly $r = (t, \dashv, f_{xy})_{x \dashv y} \in P'$ and $r \geqq p, q$.

(5.8) Corollary. **P**$'$ *has the* μ^+-c.c., *assuming* $\mu^{<\mu} = \mu$.

Proof. By the usual Δ-system argument. If $A \in [P']^{\mu^+}$ there is $A' \in [A]^{\mu^+}$ s.t. for $p, q \in A'$, $p \restriction \mu = q \restriction \mu$. This is because if $\mu^{<\mu} = \mu$ then $\operatorname{card}\{p \restriction \mu : p \in P'\} \leqq \mu$. But then there is $A'' \in [A']^{\mu^+}$ s.t. for $p, q \in A''$, S^p_μ, S^q_μ have the Δ-property. So by (5.7), the elements of A'' are pairwise compatible.

(5.9) Lemma. **P**$'$ *is* μ-*directed closed.*

Proof. Let $D \in [P']^{<\mu}$ be directed. For $p \in D$, set $\alpha(p) = \max((\operatorname{dom} t^p) \cap \mu)$, $S(p) = S^p_\mu$, $S'(p) = S'$-in-the-sense-of-$S(p)$. For $\eta \in S(p)$, δ^p_η, $f^p_\eta = \delta_\eta$, f_η in the sense of $S(p)$ and if $\eta \in X$, $\lambda^p_\eta = \lambda_\eta$ in the sense of $S(p)$. Finally, $\sigma(p) = \max S(p)$.

Note that by (5.7), we may as well assume that $A = \{\alpha(p) : p \in D\}$ has no largest element, because if $p \in D$ is such that $\alpha(p)$ is $\max A$ then we easily see, since D is directed, and by (5.7), that p is the maximum element of D.

So, let $\alpha = \sup_{p \in D} \alpha(p)$. Then $\alpha < \mu$. Let $\bar t = \bigcup_{p \in D} t^p \restriction \mu$. Define $\bar\dashv$ on $\bar t$ by: $x \bar\dashv y$ iff for some $p \in D$, $x \dashv^p y$. For $x \bar\dashv y$, let $\bar f_{xy} = f^p_{xy}$ where p is such that $x \dashv^p y$. This is clearly unambiguous, and for $p \in D$,

$$p \restriction \mu = (\bar t \restriction \alpha(p)+1, \bar\dashv \restriction (\bar t \restriction \alpha(p)+1), \bar f_{xy})_{x \bar\dashv y,\, l(y) \leqq \alpha(p)},$$

where, naturally, $\bar t \restriction \alpha(p)+1 = \{y \in \bar t : l(y) \leqq \alpha(p)\}$.

Let $\bar S = \bigcup_{p \in D} S(p)$. If $p \in D$ and $\eta \in S(p) \cap X$, set $\lambda_\eta = \sup \bar S \cap \eta$; then clearly $\lambda_\eta = \sup_{q \in D_p} \lambda^q_\eta$, where $D_p = \{q \in D : p \leqq q\}$. Let $S = \bar S \cup \{\lambda_\eta : \eta \in \bar S \cap X\} \cup \{\sigma\}$, where $\sigma = \sup \bar S$. Note that $\sigma = \sup_{p \in D} \sigma(p)$. Then, by (5.5)(b), S is the smallest acceptable set $\supseteq \bar S$. As in (5.5), let $S' = (S \cap \sigma) \setminus \{\lambda_\eta : \eta \in S \cap X\}$, and for $\eta \in S$, let $\delta_\eta = \text{o.t.}\, S' \cap \eta$, $f_\eta : \delta_\eta \to S' \cap \eta$ be the increasing enumeration; note that $S' \cap \sigma = \bigcup_{p \in D} (S'(p) \cap \sigma)$.

Set $t = \bar{t} \cup \{(\alpha, \delta_\eta): \eta \in S\} \cup \{(\mu, \eta): \eta \in S\}$. For $x, y \in t$, set $x \dashv y$ iff:

(i) $x, y \in \bar{t}$ and $x \bar{\dashv} y$; then $f_{xy} = \bar{f}_{xy}$, or

(ii) for some $p \in D$, $x, y \in t^p$, $x \dashv^p y$, $\mathrm{l}(y) = \mu$; then $f_{xy} = f^p_{xy}$, or

(iii) for some $p \in D$, some $\eta \in S(p) \cap X$, $y = (\mu, \lambda_\eta)$, $x \dashv^p y' = (\mu, \eta)$; then $f_{xy} = f^p_{xy'}$, or

(iv) for some $\eta \in S$, $x = (\alpha, \delta_\eta)$, $y = (\mu, \eta)$; then $f_{xy} = f_\eta$, or

(v) for some $\eta \in S$, $y = (\alpha, \delta_\eta)$, $\mathrm{l}(x) < \alpha$, $x \dashv y' = (\mu, \eta)$; then $f_{xy} = f^{-1}_{yy'} \circ f_{xy'}$.

(5.9.1) The first thing to check is that the conjunction of (ii), (iii) cannot lead to ambiguities if $\eta \in \bar{S} \cap X$ but $\lambda_\eta \in \bar{S}$. So, suppose $\eta \in \bar{S} \cap X$, $\lambda_\eta \in \bar{S}$. We show that for all $p \in D$ with $\lambda_\eta = \lambda^p_\eta$, $x \dashv^p (\mu, \lambda_\eta)$ iff $x \dashv^p (\mu, \eta)$ and if $x \dashv^p (\mu, \eta)$, then $f^p_{x(\mu,\eta)} = f^p_{x(\mu,\lambda_\eta)}$. Clearly if $x \dashv^p (\mu, \eta)$ then $x \dashv^p (\mu, \lambda_\eta)$ and $f^p_{x(\mu,\eta)} = f^p_{x(\mu,\lambda_\eta)}$. So, suppose $x \dashv^p (\mu, \lambda_\eta)$. But then $\delta^p_\eta = \delta^p_{\lambda_\eta}$ and, setting $y = (\alpha(p), \delta^p_{\lambda_\eta})$, $x = (\alpha(p), \delta^p_{\lambda_\eta})$ or $x \dashv^p y$, since $x \dashv^p (\mu, \lambda_\eta)$. By transitivity of $\dashv^p$, $x \dashv^p (\mu, \eta)$ since $y \dashv^p (\mu, \eta)$.

Thus, in carrying out further verifications, when $\eta \in \bar{S} \cap X$, and $\lambda_\eta \in \bar{S}$, we know that we may suppose this is in virtue of (ii), or of (iii), as best suits our purposes.

(5.9.2) Clearly $x \dashv y \Rightarrow \mathrm{l}(x) < \mathrm{l}(y)$. Let's check that $\dashv$ is a tree, and that $(f_{xy})_{x \dashv y}$ is commutative. First, let's see that $\dashv$ is transitive. Suppose $x \dashv y \dashv z$. It suffices to consider $\mathrm{l}(z) = \mu$, since if $\mathrm{l}(z) < \alpha$, this is by (i), and if $\mathrm{l}(z) = \alpha$ and transitivity is proved when $\mathrm{l}(z) = \mu$, let $z \dashv z'$ with $\mathrm{l}(z') = \mu$. Then $x \dashv z'$, and by (vi) $x \dashv z$. Similarly, we can reduce the verification that $f_{xz} = f_{yz} \circ f_{xy}$ to the case when $\mathrm{l}(z) = \mu$. So, suppose $\mathrm{l}(z) = \mu$. First, consider the case when $\mathrm{o}(z) \in \bar{S}$, say $z \in t^p$, where $p \in D$. If $\mathrm{l}(y) < \alpha$, we may suppose that $y \dashv z$ in virtue of (ii), by (5.9.1), so let $q \in D_p$ be such that $y \dashv^q z$. Then $x \dashv^q y$ and so $x \dashv^q z$, $f^q_{xz} = f^q_{yz} \circ f^q_{xy}$. If $\mathrm{l}(y) = \alpha$, the only clause which gives $y \dashv z$ is (iv), and the only clause which gives $x \dashv y$ is (v). Hence we must have $x \dashv z$ and $f_{xy} = f^{-1}_{yz} \circ f_{xz}$; i.e. $f_{xz} = f_{yz} \circ f_{xy}$. Next consider the case where $z = (\mu, \lambda_\eta)$ for some $\eta \in \bar{S} \cap X$ and $\lambda_\eta \notin \bar{S}$. If $\mathrm{l}(y) = \alpha$, the only clause which gives $y \dashv z$ is (iv), and the only clause which gives $x \dashv y$ is (v); once again, we must have $x \dashv z$ and $f_{xy} = f^{-1}_{yz} \circ f_{xz}$; i.e. $f_{xz} = f_{yz} \circ f_{xy}$. If $\mathrm{l}(y) < \alpha$, since $\lambda_\eta \notin \bar{S}$, then the only clause which gives $y \dashv z$ is (iii), and then $y \dashv z' = (\mu, \eta)$, $f_{yz} = f_{yz'}$. By what we've already proved, $x \dashv z'$ and $f_{xz'} = f_{yz'} \circ f_{xy}$; but then, by (iii), $x \dashv z$ and $f_{xz} = f_{xz'} = f_{yz'} \circ f_{xy} = f_{yz} \circ f_{xy}$. Finally, if $z = (\mu, \sigma)$ and $\sigma \notin \bar{S}$, there is nothing to verify since then $y \dashv z$ iff $y = (\alpha, \delta_\sigma)$ and in this case, y is minimal in $\dashv$.

Next, let's see the set of $\dashv$-predecessors of z is totally ordered by $\dashv$. Clearly, if $\mathrm{l}(z) = \mu$, there is exactly one $\dashv$-predecessor y of z with $\mathrm{l}(y) = \alpha$, namely, if

$z=(\mu,\eta)$, $y=(\alpha,\delta_\eta)$, and (v) then guarantees that if $\mathrm{l}(x)<\alpha$ and $x \mapsto\!\!\!3\ z$ then $x \mapsto\!\!\!3\ v$. If $\mathrm{l}(z)=\alpha$, then if $x \mapsto\!\!\!3\ z$, $y \mapsto\!\!\!3\ z$ then $\mathrm{l}(x),\mathrm{l}(y)<\alpha$, and $x \mapsto\!\!\!3\ z$, $y \mapsto\!\!\!3\ z$ in virtue of (v); so if $z=(\alpha,\delta_\eta)$, then $x \mapsto\!\!\!3\ z'$, $y \mapsto\!\!\!3\ z'$ where $z'=(\mu,\eta)$. If $\mathrm{l}(z)<\alpha$, and $x \mapsto\!\!\!3\ z$, $y \mapsto\!\!\!3\ z$, then this is in virtue of (i) and so the conclusion follows by looking inside an appropriate $p\in D$. So it suffices to verify this when $z=(\mu,\eta)$, $\mathrm{l}(x)$, $\mathrm{l}(y)<\alpha$. If $\eta=\lambda_{\eta'}$ for $\eta'\in\bar S\cap X$, and the conclusion holds for $z'=(\mu,\eta')$ then the conclusion holds for z, since, by (5.9.1), we can assume that $x \mapsto\!\!\!3\ z$, $y \mapsto\!\!\!3\ z$ in virtue of (iii). So, finally suppose that $z=(\mu,\eta)$, $\eta\in\bar S$, $x \mapsto\!\!\!3\ z$, $y \mapsto\!\!\!3\ z$. Then this holds in virtue of (ii), so since D is directed, there is $p\in D$ with $x \mapsto\!\!\!3^p z$, $y \mapsto\!\!\!3^p z$. But then clearly either $x \mapsto\!\!\!3^p y$ or $y \mapsto\!\!\!3^p x$.

This gives (5.4)(i), (5.2)(ii). By reductions as above it's easy to see that (5.2)(iii) holds. (5.2)(v′) holds for $\mathrm{l}(y)=\alpha$ by (5.9.6) below and in virtue of (iv), (v), and clearly holds, looking inside an appropriate $p\in D$, if $\mathrm{l}(y)<\alpha$. For (5.2)(vi), if $\mathrm{l}(y)=\mu$ and $\mathrm{l}(x)=\alpha$ then $\mathrm{o}(y)\in X$, and $\lambda=\lambda_\eta$ so that (5.2)(vi) holds by construction. If $\mathrm{l}(y)=\mu$ and $\mathrm{l}(x)<\alpha$, then either $x \mapsto\!\!\!3\ y$ in virtue of (iii) which means that for $\eta\in\bar S\cap X$, $y=(\mu,\lambda_\eta)$, $x \mapsto\!\!\!3\ y'=(\mu,\eta)$ in virtue of (ii), or $x \mapsto\!\!\!3\ y$ in virtue of (ii) directly. The conclusion then holds looking inside an appropriate $p\in D$. Also, the conclusion follows in this manner if $\mathrm{l}(y)<\alpha$. Finally if $\mathrm{l}(y)=\alpha$ then $x \mapsto\!\!\!3\ y$ holds in virtue of (v); going up to $y'=(\mu,\eta)$ by $f_{yy'}$ where $y=(\alpha,\delta_\eta)$ the conclusion holds for $z'=(\mu,\lambda')$ where $\lambda'=f_{yy'}(\lambda)$, by what we've already proved.

Also, we've already proved in (5.5)(c), that $z \mapsto\!\!\!3\ z'$, $f_{zz'}=f_{y'}\restriction\lambda$. By (5.9.2), $x \mapsto\!\!\!3\ z$, since $x \mapsto\!\!\!3\ z'$, $z \mapsto\!\!\!3\ z'$ and $\mathrm{l}(x)<\mathrm{l}(z)$. Also, by (5.9.2), $f_{xz'}=f_{zz'}\circ f_{xz}$; i.e. $f_{xz}=f^{-1}_{zz'}\circ f_{xz'}=f^{-1}_{zz'}\circ f_{xy'}=f^{-1}_{yy'}\circ f_{xy'}=f_{xy}$.

It remains to verify (5.2)(iv′). If $\mathrm{l}(y)<\alpha$, this is by looking inside an appropriate $p\in D$. Let's verify (5.2)(iv′) for $\mathrm{l}(y)=\mu$. First, in this case, y is non-minimal in $\mapsto\!\!\!3$, since letting $\eta=\mathrm{o}(y)$, $(\alpha,\delta_\eta)\mapsto\!\!\!3\ y$. Second, if $\eta=\sigma\notin\bar S$ then $\{\alpha\}$ is a co-initial segment of $(\operatorname{dom} t)\cap\mu$. Third, suppose $\eta=\lambda_{\eta'}$ for some $\eta'\in\bar S\cap X$, and set $y'=(\mu,\eta')$. If we know the conclusion for y', we know it for y since, by (iii), (iv) and the fact that $\delta_\eta=\delta_{\eta'}$, the $\mapsto\!\!\!3$-predecessors of y are exactly the $\mapsto\!\!\!3$-predecessors of y'. So, we verify the conclusion for $\eta\in\bar S$. Suppose $x \mapsto\!\!\!3\ y$, $\mathrm{l}(x)<\alpha'<\alpha$. Then $x \mapsto\!\!\!3\ y$ in virtue of (ii) and so there's $p\in D$ with $x \mapsto\!\!\!3^p y$, $\alpha'\leqq\alpha(p)$. Then by (5.2)(iv′) for p, there is $x \mapsto\!\!\!3^p x' \mapsto\!\!\!3^p y$ with $\mathrm{l}(x')=\alpha'$. Finally, let's verify the conclusion for $\mathrm{l}(y)=\alpha$. First consider the possibility that $y=(\alpha,\delta_\sigma)$ and $\sigma\notin\bar S$. Then y is $\mapsto\!\!\!3$-minimal, but since δ_σ is then the largest element of γ_α, nothing is required by (5.2)(iv′). Otherwise, for some $\eta\in\bar S$, $y=(\alpha,\delta_\eta)$ and the conclusion carries down to y from $y'=(\mu,\eta)$. Thus, $r=(t,\mapsto\!\!\!3,f_{xy})_{x\to y}\in P$. By construction $S=S'_\mu$ is acceptable, and r is strongly acceptable. Then clearly r is an upper bound for D. We now finish by proving the technical details needed to prove (5.2)(iv′) for $l(y) = \alpha$.

If $p \in D$, $\eta \in S(p)$, let $R_\eta = \bigcup_{q \in D_p} \operatorname{range} f^q$. If $\eta \in X$, and $\lambda_\eta \notin \bar{S}$, let $R_{\lambda_\eta} = R_\eta$. If $\sigma \notin \bar{S}$, set $R_\sigma = S' \cap \sigma$.

(5.9.3) PROPOSITION. (a) *If $\eta \in \bar{S} \cap X$ and $\lambda_\eta \in \bar{S}$, then for some $p \in D$, λ_η, $\eta \in S(p)$ and whenever $q \in D_p$, $\lambda_\eta = \lambda^q_\eta$*; (b) *thus, even here, $R_{\lambda_\eta} = R_\eta$.*

PROOF. For (a), take any $p \in D$ with λ_η, $\eta \in S(p)$ (such exists since D is directed and by hypothesis $\lambda_\eta, \eta \in \bar{S}$). Note that for $q \in D_p$, $\lambda^q_\eta \leqq \lambda_\eta$, so we can't have $\lambda_\eta \in \operatorname{range} f^q_\eta$. But then, for a unique $\eta' \in S(q) \cap X$, $\lambda_\eta = \lambda^q_{\eta'}$ (otherwise, we'd have $\lambda_\eta \in S'(q)$ and then since $\lambda_\eta < \eta$, $\lambda_\eta \in \operatorname{range} f^q_\eta$). In fact, this η' must be η, since $\eta' < \eta$ is absurd (because $\eta' > \lambda^q_{\eta'} = \lambda_\eta \geqq \lambda^q_\eta$) and $\eta < \eta'$ is again absurd (because $\eta > \lambda_\eta = \lambda^q_{\eta'}$).

For (b) first note that by (a), $\lambda_\eta \notin R_\eta$ because if so then for some $p \in D$, $\lambda_\eta \in S(p) \subseteq \bar{S}$, and (a) guarantees that for no $q \in D_p$ is $\lambda_\eta \in \operatorname{range} f^q_\eta$. If $\eta' \in R_{\lambda_\eta}$, let $p \in D$ be such that $\lambda_\eta, \eta \in S(p)$ and $\eta' \in \operatorname{range} f^p_{\lambda_\eta}$, and $\lambda_\eta = \lambda^p_\eta$ (once again, such exists since $\lambda_\eta, \eta \in S$, D is directed, and by (a)). Then $f^p_\eta = f^p_{\lambda_\eta}$ and so $\eta' \in \operatorname{range} f^p_\eta$. Starting with $\eta' \in R_\eta$, the same argument makes it clear that $\eta' \in R_{\lambda_\eta}$.

(5.9.4) PROPOSITION. *For $\eta \in \bar{S} \cap X$, $\eta' \in S$, $\lambda_\eta \notin R_{\eta'}$.*

PROOF. This is clear if $\eta' < \eta$, or if $\eta' = \sigma$ and $\sigma \notin \bar{S}$. If $\eta' = \eta$, this is by (5.9.3). If $\eta' > \eta$ and $\eta' \in \bar{S}$, choose $p \in D$ with $\eta', \eta \in S(p)$. If $\lambda_\eta \in \operatorname{range} f^p_{\eta'}$, then $\lambda_\eta \in \operatorname{range} f^p_\eta$, which, by (5.9.3), is impossible. Finally, suppose $\eta' > \eta$, $\eta' \notin \bar{S}$ and for some $\eta'' \in \bar{S} \cap X$, $\eta' = \lambda_{\eta''}$. Then, by (5.9.3), $R_{\eta'} = R_{\eta''}$, and by what we've already proved, $\lambda_\eta \notin R_{\eta''}$.

(5.9.5) PROPOSITION. *If $\sigma \in \bar{S}$, then $R_\sigma = S' \cap \sigma$.*

PROOF. By (5.9.4), $S' \cap \sigma \subseteq R_\sigma$. If $\eta' \in S' \cap \sigma$, then clearly $\eta' \in \bar{S}$. So, let $p \in D$ with $\sigma, \eta' \in S(p)$. Then clearly $\sigma = \sigma(p)$ and if $q \in D_p$, $\sigma = \sigma(q)$. Hence, we may suppose, without loss of generality, that $\eta' \in S'(p)$, since as we've noted, $S' \cap \sigma = \bigcup_{q \in D} (S'(q) \cap \sigma)$. But then $\eta' \in \operatorname{range} f^p_\sigma$.

(5.9.6) PROPOSITION. *For $\eta \in S$, $R_\eta = S' \cap \eta$.*

PROOF. This is proved if $\eta = \sigma$, so suppose $\eta < \sigma$. By (5.9.4), $R_\eta \subseteq S' \cap \eta$. Suppose $\eta' \in S' \cap \eta$. If $\eta \in \bar{S}$, once again, we find $p \in D$ with $\eta' \in S'(p)$, $\eta \in S(p)$ and, as in (5.9.5), $\eta' \in \operatorname{range} f^p_\eta$. If $\eta \notin \bar{S}$ let $\eta'' \in \bar{S} \cap X$ be such that $\eta = \lambda_{\eta''}$ (since $\eta < \sigma$). Then $R_\eta = R_{\eta''}$ by definition, and by what we've already proved, $\eta' \in R_{\eta''}$.

(5.10) PROPOSITION (Extension Property). *If* $p \in P'$, $\alpha < \mu$, $\xi < \mu^+$, *then there is* $q \in P'$, $q \geqq p$, *such that* $\xi \in S^q_\mu$, $\alpha \in \operatorname{dom} t^q$.

PROOF. Let S be acceptable, $S \supseteq S^p_\mu \cup \{\xi\}$. First obtain acceptable $q' \geqq p$ with $S^{q'}_\mu = S$ and $t^{q'} \restriction \mu = t^p \restriction \mu \cup \{(\alpha'+1, \delta_\eta): \eta \in S\}$, where $\alpha' = \max((\operatorname{dom} t^p) \cap \mu)$. Then, if $\alpha'+1 < \alpha$, obtain q by setting $t^q = t^{q'} \cup \{(\alpha'', \delta_\eta): \alpha + 1 < \alpha'' \leqq \alpha,\ \eta \in S\}$ and for $\eta \in S$, $\alpha'+1 < \alpha'' \leqq \alpha$, $(\alpha'+1, \delta_\eta) \dashv^q (\alpha'', \delta_\eta) \dashv^q (\mu, \eta)$, $f^q_{(\alpha'+1,\delta_\eta),(\alpha'',\delta_\eta)} = \mathrm{id} \restriction \delta_\eta$, $f^q_{(\alpha'',\delta_\eta),(\mu,\eta)} = f^{q'}_{(\alpha'+1,\delta_\eta),(\mu,\eta)}$, and, if $\eta \in X$, the same holds with λ_η replacing η. Of course the two operations could have been performed in opposite order first getting q' with $\alpha \in \operatorname{dom} t^{q'}$, and then getting q with $\eta \in S^q_\mu$.

(5.11) Now let's see that $\mathbf{P}'$ is μ-special. We've already defined X and $\tilde{X}$ in (5.5). Let $(x_\alpha : \alpha < \mu^+)$ increasingly enumerate X, and for $s \in [\mu^+]^{<\mu}$, let $X_s = \{0\} \cup \{x_{s(i)}: i < \lg s\}$, and let $\tilde{X}_s = X_s \cup (X_s)^*$. Let $\bar{s} = \mathrm{id} \restriction \lg s$, let $\sigma_s \restriction \tilde{X}_{\bar{s}}$ be the unique order-preserving map from $\tilde{X}_{\bar{s}}$ onto $\tilde{X}_s$ and for $\beta < \mu$, $\mu . \alpha \in \tilde{X}_{\bar{s}}$, let $\sigma_s((\mu . \alpha) + \beta) = \sigma_s(\mu . \alpha) + \beta$.

Let $\mathcal{T} = \{\bar{p} \in P' : X \cap S^{\bar{p}}_\mu$ is an initial segment of $X\}$. Thus, *if* $\mu^{<\mu} = \mu$, $\operatorname{card} \mathcal{T} = \mu$. Let $\lg \bar{p} = \mathrm{o.t.}(S^{\bar{p}}_\mu \cap X)$. If $s \in [\mu^+]^{\lg p}$ define $p = \bar{p}(s) \in P'$ by $p \restriction \mu = \bar{p} \restriction \mu$, $S^p_\mu = \sigma''_s S^{\bar{p}}_\mu$; thus, if $\eta \in S^{\bar{p}}_\mu \cap \mu$, $y = (\mu, \eta)$, $x \dashv^{\bar{p}} y$, then $x \dashv^p y$ and $f^p_{xy} = f^{\bar{p}}_{xy}$, while if $\eta \in S^{\bar{p}}_\mu \setminus \mu$, $y = (\mu, \eta)$, $x \dashv^{\bar{p}} y$, then, setting $y' = (\mu, \sigma_s(\eta))$, $x \dashv^p y'$ and $f^p_{xy'} = \sigma_s \circ f^{\bar{p}}_{xy}$.

If $p \in P'$, let $J = J^p = \{j < \mu^+ : x_j \in S^p_\mu \cap X\}$ and let $s = s^p$ be the increasing enumeration of J. Obtain $\bar{p} \in \mathcal{T}$ by reversing the above construction. Then clearly $p = \bar{p}(s)$. It's now clear that $\mathbf{P}'$ is μ-special.

(5.12) We now see that $\mathbf{P}' \in \mathcal{S}_\mu$. The Indiscernibility property is obvious. The Amalgamation property is by (5.8). So we have:

LEMMA. $\mathbf{P}' \in \mathcal{S}_\mu$.

(5.13) Let $i < \mu$, $\bar{p} \in \mathcal{T}$. Set $\bar{p} \in D_i$ iff $i \in \operatorname{dom} t^{\bar{p}}$, and if $\eta \in \tilde{X}_{\mathrm{id} \restriction \lg \bar{p}}$, then $\eta + i \in S^{\bar{p}}_\mu$.

Now suppose G meets all of the $D^{\#}_i$ uniformly. Let $(T, \dashv) = \bigcup_{p \in G} (t^p, \dashv^p)$ and for $p \in G$, $x, y \in t^p$, if $x \dashv^p y$, let $f_{xy} = f^p_{xy}$. The only premorass property that requires verification is that if $y \in T$, $\mathrm{l}(y) = \mu$, then $\mathrm{o}(y) = \bigcup_{x \dashv y} \operatorname{range} f_{xy}$. So let $\eta < \mathrm{o}(y)$. Since G meets all the $D^{\#}_i$ uniformly, there is $p \in G$ such that $\eta, \eta + 1$, $\mathrm{o}(y) \in S^p_\mu$. Let $\alpha = \max((\operatorname{dom} t^p) \cap \mu)$, and let $x = (\alpha, \delta^p_{\mathrm{o}(y)})$, so that $x \dashv^p y$. Since $\eta + 1 \in S^p_\mu$, for no $\eta' \in S^p_\mu$ is $\eta = \lambda_{\eta'}$. Thus $\eta \in \operatorname{range} f^p_{xy} = f_{xy}$.

We now clearly have:

THEOREM 1 (left-to-right). *$S_\mu \Rightarrow$ there's a $(\mu, 1)$-morass.*

PROOF. By the immediately preceding material, if S_μ then there's a $(\mu, 1)$-premorass. Hence, by (5.3), there is a $(\mu, 1)$-morass.

§6. The main theorem

This section is devoted to proving the main theorem:

(6.1) THEOREM 1 (right-to-left). *For all regular $\mu \geqq \aleph_1$ if there is a $(\mu, 1)$-morass, then S_μ.*

The proof is spread out over the remainder of this section. The basic set-up of the proof has not been modified, but in order to avoid the Restriction Property, we have incorporated, with his permission, Velleman's idea of "spreading-out" the ranges of the $f_{\bar\nu\nu}$ by left-multiplying by "gap ordinals" (the γ_α of (6.2)); this also necessitated the introduction of the $\rho(\alpha)$.

As noted in the Introduction, this theorem was formulated by Shelah, who suggested that a proof along the lines of the proof of the two-gap–two-cardinal theorem in L might work; the theorem was proved by Stanley.

Throughout the rest of this section, assume that $\mathcal{M} = (\mathcal{S}, S^0, S^1, \prec, \pi_{\bar\nu\nu})_{\bar\nu \prec \nu}$ is a $(\mu, 1)$-morass. Assume that $\mathbf{P} \in \mathcal{S}_\mu$, and that $(D_i : i < \mu)$ is a family of dense subsets of $\mathcal{T}$.

(6.2) For $\alpha \in S^0$, let $\theta(\alpha) = \sup S_\alpha$, if S_α has no largest element, while if S_α has ν^* as largest element, set $\theta(\alpha) = \nu^* + \alpha$. For $\nu \in S_\alpha$, set $\theta(\nu) = \theta(\alpha)$. $\xi' - \xi$ denotes ordinal subtraction: if $\xi < \xi'$ then $\xi' - \xi$ is the unique ζ such that $\xi + \zeta = \xi'$. We shall also set $\delta(\nu) = \nu + \alpha$. Note that $\theta(\alpha) = \sup\{\delta(\nu) : \nu \in S_\alpha\}$, and that if $\alpha < \alpha'$, then $\theta(\alpha) < \alpha'$.

In what follows we shall assume that the tree maps $\pi_{\bar\nu\nu}$ preserve ordinal arithmetic, i.e. if $\bar\xi, \bar\zeta < \bar\nu$, and $\xi = \pi_{\bar\nu\nu}(\bar\xi)$, $\zeta = \pi_{\bar\nu\nu}(\bar\zeta)$ then $\xi + \zeta = \pi_{\bar\nu\nu}(\bar\xi + \bar\zeta)$, $\xi \cdot \zeta = \pi_{\bar\nu\nu}(\bar\xi \cdot \bar\zeta)$ (note: this makes sense, by the p.r. closure of $\bar\nu$, ν). If the morass $\mathcal{M}$ does not have this property, it can be "thinned" to obtain an $\mathcal{M}'$ which does (cf. [12]).

If $1 \leqq \bar\gamma \leqq \gamma < \mu$, $\bar\nu \prec \nu$, let $\bar\alpha = \alpha_{\bar\nu}$, $\alpha = \alpha_\nu$, and suppose $\bar\rho$ is such that $\bar\gamma . \theta(\bar\nu) \leqq \bar\rho < \gamma \cdot \alpha$. We shall define $f_{\bar\nu\nu}^{\bar\gamma,\gamma,\bar\rho} : \bar\rho \to \gamma \cdot \delta(\nu)$ as follows. First, suppose $\bar\xi < \bar\nu$, $\zeta < \gamma$; let $\xi = \pi_{\bar\nu\nu}(\bar\xi)$:

(1) $f_{\bar\nu\nu}^{\bar\gamma,\gamma,\bar\rho}((\bar\gamma \cdot \bar\xi) + \zeta) = (\gamma \cdot \xi) + \zeta$.

Also, if $\xi < \bar\rho - (\bar\gamma \cdot \bar\nu)$ (ordinal subtraction), then:

(2) $f_{\bar\nu,\nu}^{\bar\gamma,\gamma,\bar\rho}((\bar\gamma \cdot \bar\nu) + \xi) = (\gamma \cdot \nu) + \xi$.

Our approach is as follows. We'll define by induction on $\alpha \in S^0 \cap \mu$, conditions $p(\alpha) \in P$, and ordinals $\gamma(\alpha), \rho(\alpha) < \mu$ with the following properties.

(3) $(p(\alpha) : \alpha \in S^0 \cap \mu)$, $(\gamma(\alpha) : \alpha \in S^0 \cap \mu)$, $(\rho(\alpha) : \alpha \in S^0 \cap \mu)$ are increasing; $(\gamma(\alpha) : \alpha \in S^0 \cap \mu)$ is continuous; we'll also set $\gamma(\mu) = \mu$.

(4) $p(\alpha)$ has the form $\bar{p}(\alpha)\,(\mathrm{id} \restriction \rho(\alpha))$, where $\bar{p}(\alpha) \in \mathscr{T}$; thus $\lg \bar{p}(\alpha) = \rho(\alpha)$.

(5) If $\beta, \alpha \in S^0 \cap \mu$, and $\beta < \alpha$ then $\gamma(\beta) \cdot \theta(\beta) \leqq \rho(\beta) < \gamma(\alpha) \cdot \alpha$.

(6) If $\bar{\alpha} \in S^0$, $\bar{\nu} \in S_{\bar{\alpha}}$, $\nu \in S_\alpha$ and $\bar{\nu} \dashv \nu$, let $\bar{\gamma} = \gamma(\bar{\alpha})$, $\gamma = \gamma(\alpha)$, $\bar{\rho} = \rho(\bar{\alpha})$.

We then define $p(\bar{\nu}, \nu) = \bar{p}(\alpha)\,(f^{\bar{\gamma},\gamma,\bar{\rho}}_{\bar{\nu},\nu})$ (note that this makes sense, since by (5) we are in the presence of the hypotheses used to define $f^{\bar{\gamma},\gamma,\rho}_{\bar{\nu},\nu}$). We shall abuse notation by setting $f_{\bar{\nu}\nu} = f^{\bar{\gamma},\gamma\bar{\rho}}_{\bar{\nu},\nu}$, and *taking it to be defined once we have defined* $p(\bar{\alpha}), \gamma(\bar{\alpha}), \rho(\bar{\alpha})$, *even though, formally, the definition depends on* $\gamma = \gamma(\alpha)$, *and this will not be defined until later.*

(7) Under the hypotheses of (6) and using the notation from (6), we shall require that if $\alpha < \mu$, then $p(\bar{\nu}, \nu) \leqq p(\alpha)$.

(6.3) We prove some technical lemmas which will serve in the construction. The reader may, if he's thus inclined, skip ahead to (6.4) and refer back to these lemmas when they arise. The first lemma is a basic consequence of the Indiscernibility Property and will be used again and again in what follows, in various circumstances.

The second lemma is pure "morass-theory", and will be used in case (E2) of (6.6), rather than including this rather intricate morass argument in the construction.

(6.3.1) Lemma. *Suppose* $p = \tau(s)$, $p' = \tau'(s') \in P$, *suppose* $p \leqq p'$, *and suppose that* $s' = \mathrm{id} \restriction \lg s'$. *Let* $g : \lg s \to \lg s'$ *be increasing and such that* $s = s' \circ g$. *Then*:

(a) $g = s$;

(b) if h *is an increasing function with* $\operatorname{dom} h = \lg s'$, *then* $\tau(h \circ s) \leqq \tau'(h)$.

Proof. (a) is clear, and (b) follows readily from (a) and the Indiscernibility Property.

(6.3.2) Suppose that $\bar{\nu}$ is the immediate $\dashv$-predecessor of ν, that ν is a limit in S_{α_ν} (and hence $\bar{\nu}$ is a limit in $S_{\alpha_{\bar{\nu}}}$), and that range $\pi_{\bar{\nu}\nu}$ is cofinal in ν (thus, we're in the situation of (M7), the second continuity property). Let $(\bar{\eta}_i : i < \lambda)$ increasingly enumerate $S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}$. For $i < \lambda$, define η_i as follows:

(1) Let $\tau_i = \pi_{\bar{\nu}\nu}(\bar{\eta}_i)$; if $i = 0$, or i is a successor, let $\eta_i = \tau_i$, while if $i \in \mathrm{Lim}$, let $\eta_i = \sup \pi''_{\bar{\nu}\nu}\bar{\eta}_i$; thus $\eta_i \leqq \tau_i$, but if $i \in \mathrm{Lim}$, possibly $\eta_i < \tau_i$. In any case, we have, by (M2), and, if $i \in \mathrm{Lim}$ and $\eta_i < \tau_i$ by (M6):

(2) $\bar{\eta}_i \mapsto \eta_i$, $\pi_{\bar{\eta}_i\eta_i} = \pi_{\bar{\eta}_i\tau_i} = \pi_{\bar{\nu}\nu} \mid \bar{\eta}_i$.
It's also clear that

(3) if $i < j$ then $\tau_i < \eta_j$; further $\{\eta_i : i < \lambda\}$ is cofinal in ν. Now define η_i^*, τ_i^*, α_i, α'_i:

(4) η_i^* is the least $\mapsto$-predecessor of η_i which is $> \bar{\eta}_i$; τ_i^* is the least $\mapsto$-predecessor of τ_i which is $> \bar{\eta}_i$; $\alpha_i = \alpha_{\eta_i^*}$, $\alpha'_i = \alpha_{\tau_i^*}$,

(M7) then asserts that $\bigcup_{i<\lambda} \alpha'_i = \alpha_\nu$. It is clear that the α'_i are increasing; this can be seen using:

(5) if $i < j$, then $\tau_i \in \operatorname{range} \pi_{\tau_j^*\tau_j}$; hence there is $\tau'_i \in S_{\alpha'_j} \cap \tau_j^*$ such that $\tau_i^* \mapsto \tau'_i \mapsto \tau_i$. That $\tau_i \in \operatorname{range} \pi_{\tau_j^*\tau_j}$ is clear, since $\tau_i \in \operatorname{range} \pi_{\bar{\nu}\nu}{}''\bar{\eta}_j = \operatorname{range} \pi_{\bar{\eta}_j\tau_j}$, and range $\pi_{\bar{\eta}_j\tau_j} \subseteq \operatorname{range} \pi_{\tau_j^*\tau_j}$. Now τ'_i may be (and must be) taken to be $\pi_{\tau_j^*\tau_j}{}^{-1}(\tau_i)$, by (M2). We denote this τ'_1 as $\tau_{i,j}^*$. But now it's clear that $\alpha'_i \leqq \alpha'_j$, since τ_i^* was taken to be the least $\mapsto$-predecessor of τ_i which is $> \bar{\eta}_i$; thus $\tau_i^* \mapsto \tau_{i,j}^*$, or $\tau_i^* = \tau_{i,j}^*$. We'll see that $\tau_i^* \neq \tau_{i,j}^*$, which will then guarantee that $\alpha'_i < \alpha'_j$. Note that $\tau_{i,j}^*$ is a limit in $\mapsto$ (by (M4), since $\tau_{i,j}^* < \tau_j^*$, and $\tau_{i,j}^* \in S_{\alpha'_j}$). But by definition τ_i^* is an immediate successor of $\bar{\eta}_i$ in $\mapsto$. We now show that similar conclusions hold for the η_i, α_i.

LEMMA. (a) *If $i < j$ there's $\eta_{i,j}^* \in S_{\alpha_j} \cap \eta_j^*$ such that $\eta_i^* \mapsto \eta_{i,j}^{*\prime} \mapsto \eta_i$.*
(b) *The α_i's are increasing, continuous and $\bigcup_{i<\lambda} \alpha_i = \alpha_\nu$.*

PROOF. (a) We first note that, as in (5), $\tau_i \in \operatorname{range} \pi_{\eta_j^*\eta_j}$, since range $\pi_{\bar{\eta}_j\eta_j} \subseteq$ range $\pi_{\eta_j^*\eta_j}$, and $\pi_{\bar{\eta}_j\eta_j} = \pi_{\eta_j\tau_j} = \pi_{\bar{\nu}\nu} \mid \bar{\eta}_j$. Let τ' be such that $\pi_{\eta_j^*\eta_j}(\tau') = \tau_i$. As usual, $\tau' \mapsto \tau_i$, $\tau' \in S_{\alpha_j} \cap \eta_j^*$, $\pi_{\tau'\tau_i} = \pi_{\eta_j^*\eta_j} \mid \tau'$.

If $\tau_i = \eta_i$, then we conclude as in (5), and as in (5), we conclude that α_i $(= \alpha'_i) < \alpha_j$. So, suppose that $i \in \mathrm{Lim}$, that $\eta_i = \sup \pi_{\bar{\nu}\nu}{}''\bar{\eta}_i = \sup \pi_{\bar{\eta}_i\tau_i}{}''\bar{\eta}_i < \tau_i$. Let $K = \{k < i : k = 0 \text{ or } k \text{ is a successor}\}$. Thus $\eta_i = \bigcup_{k\in K} \tau_k = \bigcup_{k\in K} \eta_k$, and, by what we've already argued, for $k \in K$, η_k $(= \tau_k) \in \operatorname{range} \pi_{\eta_j^*\eta_j}$. For $k \in K$, let $\eta'_k = \pi_{\eta_j^*\eta_j}^{-1}(\eta_k)$, so that $\eta'_k \mapsto \eta_k$, $\pi_{\eta'_k\eta_k} = \pi_{\eta_j^*\eta_j} \mid \eta'_k$. Let $\eta' = \bigcup_{i\in K} \eta'_k$. Thus $\eta' \leqq \tau'$. Let $\eta'' = \pi_{\eta_j^*\eta_j}(\eta'')$. Thus $\eta'' \in S_{\alpha_\nu}$ and $\eta_i \leqq \eta'' \leqq \tau_i$. Also, as usual $\eta' \mapsto \eta''$, $\pi_{\eta'\eta''} = \pi_{\eta_j^*\eta_j} \mid \eta'$. Thus $\pi_{\eta'\eta''}(\eta'_k) = \eta_k$, for $k \in K$. But then sup range $\pi_{\eta'\eta''} = \bigcup_{k\in K} \eta_k = \eta_i$. Thus, by (M6) applied to η', η'', $\eta' \mapsto \eta_i$, $\pi_{\eta'\eta_i} = \pi_{\eta'\eta''}$. We take $\eta_{i,j}^* = \eta'$.

(b) From (a), we conclude as in (5) that the α_i's are increasing. Their continuity follows from (M7).

That $\bigcup_{i<\lambda} \alpha_i = \alpha_\nu$ follows from the fact, remarked in the proof of (a), that if $i = 0$ or i is a successor ordinal, then $\alpha_i = \alpha'_i$, and the fact that such i are cofinal in λ.

(6.4) We return to the main thread of the argument. Before actually giving the construction, in (6.6), we shall derive some consequences of the properties (1)–(7) of (6.2), which we shall be carrying as inductive hypotheses. We also develop additional aspects of the construction which can be removed from the induction.

(6.4.1) PROPOSITION. *Suppose $\alpha \in S^0$ and that* (1)–(7) *of* (6.2) *hold for all* $\alpha' \in S^0 \cap \alpha$.

(a) *If $\nu \in S_\alpha$, if $\bar{\nu} \dashv\!\!\!\dashv \nu' \dashv\!\!\!\dashv \nu$, then $p(\bar{\nu}, \nu) \leqq p(\nu', \nu)$.*

(b) *If $\nu, \tau \in S_\alpha$, $\nu < \tau$, if $\bar{\nu} \dashv\!\!\!\dashv \nu$, if $\tau' \dashv\!\!\!\dashv \tau$, if $\nu \in \operatorname{range} \pi_{\tau'\tau}$ and if $\alpha_{\tau'} > \alpha_{\bar{\nu}}$, then $p(\bar{\nu}, \nu) \leqq p(\tau', \tau)$.*

(c) *Let $S'_\alpha = \{\nu \in S_\alpha : \nu$ is a limit in $\dashv\!\!\!\dashv\}$, and suppose $S'_\alpha \neq \emptyset$. Thus $S_\alpha = S'_\alpha$, or $S_\alpha = S'_\alpha \cup \{\max S_\alpha\}$, and $\max S_\alpha$ is not a limit in $\dashv\!\!\!\dashv$. Then $D = \{p(\bar{\nu}, \nu) : \bar{\nu} \dashv\!\!\!\dashv \nu, \nu \in S'_\alpha\}$ is directed.*

PROOF. (a) We use (6.3.1); we let $p = p(\bar{\nu}, \nu')$, $p' = p(\alpha_{\nu'})$, and we let $h = f_{\nu'\nu}$; we need only remark that $f_{\bar{\nu}\nu} = f_{\nu'\nu} \circ f_{\bar{\nu}\nu'}$.

(b) We let $\nu' = \pi^{-1}_{\tau'\tau}(\nu)$, and, once again, we use (6.3.1); we let $p = p(\bar{\nu}, \nu')$, and $p' = p(\alpha_{\nu'})$ $(= p(\alpha_{\tau'}))$; we let $h = f_{\tau'\tau}$. This time, it's slightly more work to conclude that $f_{\bar{\nu}\nu} = f_{\tau'\tau} \circ f_{\bar{\nu}\nu'}$. We use the facts that $f_{\bar{\nu}\nu} = f_{\nu'\nu} \circ f_{\bar{\nu}\nu'}$, that range $f_{\bar{\nu}\nu'} \subseteq \gamma(\alpha_{\nu'}) \cdot \delta(\nu')$, and that, since $\delta(\nu') = \nu' + \alpha_{\nu'}$, since $\pi_{\tau'\tau} \restriction \alpha_{\nu'} = \mathrm{id} \restriction \alpha_{\nu'}$, and since $\pi_{\tau'\tau}$ preserves ordinal arithmetic, $f_{\nu'\nu} = f_{\tau'\tau} \restriction \gamma(\alpha_{\nu'}) \cdot \delta(\nu')$.

(c) This now follows from (a) and (b), since suppose $\nu < \tau$, $\nu, \tau \in S'_\alpha$, $\bar{\nu} \dashv\!\!\!\dashv \nu$, $\bar{\tau} \dashv\!\!\!\dashv \tau$. Since τ is a limit in $\dashv\!\!\!\dashv$, we find $\tau' \dashv\!\!\!\dashv \tau$ with $\nu \in \operatorname{range} \pi_{\tau'\tau}$, $\alpha_{\tau'} \geqq \alpha_{\bar{\tau}}$, $\alpha_{\tau'} > \alpha_{\bar{\nu}}$. But then $p(\tau', \tau) \in D$ and is a common extension of $p(\bar{\nu}, \nu)$, $p(\bar{\tau}, \tau)$.

(6.4.2) In order to obtain an ideal meeting all the $D^{\#}_i$ uniformly, let $(X_i : i < \mu)$ be a partition of μ into μ stationary sets. Having defined $\gamma(\alpha)$, our strategy for constructing $p(\alpha)$, $\rho(\alpha)$ will be as follows: we'll first find an auxiliary condition $p^*(\alpha)$ with the following properties:

(1′) $p^*(\alpha) = \bar{p}^*(\alpha)$ $(\mathrm{id} \restriction \bar{\rho}(\alpha))$, where $\bar{p}^* \in \mathcal{T}$, $\gamma(\alpha) \cdot \theta(\alpha) \leqq \bar{\rho}(\alpha)$,

(2′) if $\alpha' \in S^0 \cap \mu$ then $p(\alpha') \leqq p^*(\alpha)$,

(3′) if $\eta \in S_\alpha$, $\bar{\eta} \dashv\!\!\!\dashv \eta$, then $p(\bar{\eta}, \eta) \leqq p^*(\alpha)$.

We then let $i < \mu$ be such that $\alpha \in X_i$. We choose $\bar{p}(\alpha) \geqq \bar{p}^*(\alpha)$, $\bar{p}(\alpha) \in D_i$, and we let $\rho(\alpha) = \lg \bar{p}(\alpha)$ (so $\rho(\alpha) \geqq \bar{\rho}(\alpha)$). Then $p(\alpha) = \bar{p}(\alpha)$ $(\mathrm{id} \restriction \rho(\alpha)) \geqq p^*(\alpha)$.

(6.4.3) We describe briefly the way we shall obtain $\gamma(\alpha)$. If $\alpha = \inf S^0$, $\gamma(\alpha) = 1$. If α is a limit point in S^0, we take $\gamma(\alpha) = \sup_{\beta \in S^0 \cap \alpha} \gamma(\beta)$. Clearly property (5) of (6.2) is preserved. If α is an immediate successor in S^0, say of β, we take $\gamma(\alpha)$ to be the least $\gamma > \gamma(\beta)$ such that $\gamma \cdot \alpha > \gamma(\beta) \cdot \rho(\beta)$.

(6.5) We now verify that if the construction is carried out with properties (1)–(7) of (6.2), and following the recipe of (6.4.2), we obtain an ideal meeting all the D_i^* uniformly (in what follows we'll say a sufficiently-generic G).

(6.5.1) DEFINITION. Let

$$G = \{p \in P : (\exists \nu \in S_\mu)(\exists \bar{\nu})(\bar{\nu} \dashv\!\!\!\dashv \nu \text{ and } p \leqq p(\bar{\nu}, \nu))\}.$$

(6.5.2) LEMMA. *G is sufficiently generic and μ-complete.*

PROOF. By construction G is downward closed. By (6.4.1)(c) the set $D = \{p(\bar{\nu}, \nu) : \bar{\nu} \dashv\!\!\!\dashv \nu,\ \nu \in S_\mu\}$ which generates G is directed. Hence so is G.

For sufficient genericity, let $i < \mu$, let $s \in [\mu^+]^{<\mu}$. The crucial observation is that we can find $\nu \in S_\mu$ and $\bar{\nu} \dashv\!\!\!\dashv \nu$ with range $s \subseteq \text{range} f_{\bar{\nu}\nu}$ and with $\alpha_{\bar{\nu}} \in X_i$. This is by (M3)–(M5). Let $\bar{\alpha} = \alpha_{\bar{\nu}}$. Then adopting the notation of (6.4.2), $\bar{p}(\bar{\alpha}) \in D_i$, and $\bar{p}(\bar{\alpha})(f_{\bar{\nu}\nu}) \in G$.

That G is μ-complete is seen as follows: it clearly suffices to see that D is μ-directed, i.e. if $D' \in [D]^{<\mu}$ then there's $p \in D$ with $p' \leqq p$ for all $p' \in D'$. So let $D' = \{p(\bar{\nu}^i, \nu^i) : i < \beta\}$, where $\beta < \mu$. We can clearly find $\nu \in S_\mu$ with $\nu \geqq \bigcup_{i<\beta} \nu^i$, and $\nu' \dashv\!\!\!\dashv \nu$ such that for all $i < \beta$, $\nu^i \in \text{range}\, \pi_{\nu'\nu}$ and $\alpha_{\nu'} > \alpha_{\bar{\nu}^i}$. But then, by (6.4.1)(b), $p(\nu', \nu) \geqq p(\bar{\nu}^i, \nu^i)$, and $p(\nu', \nu) \in G$.

(6.6) We now carry out the inductive construction of the $p(\alpha)$, $\gamma(\alpha)$, $\rho(\alpha)$. Following (6.4.2), (6.4.3), we shall content ourselves with constructing $\bar{p}^*(\alpha)$, since $\gamma(\alpha)$ is obtained via (6.4.3), and $\bar{p}(\alpha)$, $\rho(\alpha)$ are then obtained as in (6.4.2). Recall that we work by induction on $\alpha \in S^0 \cap \mu$, and that we are carrying properties (1)–(7) of (6.2), and their consequences, Proposition (6.4.1) (a), (b), (c) as induction hypotheses. The verifications of (1′), (2′), (3′) of (6.4.2) for $\bar{p}^*(\alpha)$ are left to the reader. Once made, these verifications mean that (1)–(7) of (6.2) will be preserved. As in (6.4.1)(c), let $S'_\alpha = \{\nu \in S_\alpha : \nu \text{ is a limit in } \dashv\!\!\!\dashv\}$. We distinguish the following cases:

(A) $S'_\alpha = \emptyset$; say $S_\alpha = \{\nu\}$.

(A1) ν is minimal in $\dashv\!\!\!\dashv$,

(A2) ν immediately succeeds $\bar{\nu}$ in $\dashv\!\!\!\dashv$.

(B) $S_\alpha = S'_\alpha \neq \emptyset$.

In all remaining cases $S'_\alpha \neq \emptyset$, $S_\alpha = S'_\alpha \cup \{\nu\}$.

(C) ν is minimal in $\dashv\!\!\!\dashv$.

(D) ν is an immediate successor of $\bar{\nu}$ in $\dashv\!\!\!\dashv$, and the immediate successor of τ in S_α (so $\tau = \max S'_\alpha$).

(E) ν is an immediate successor of $\bar{\nu}$ in $\dashv$, and a limit point of S_α:

(E1) $\pi_{\bar{\nu}\nu}$ is not cofinal in ν,

(E2) $\pi_{\bar{\nu}\nu}$ is cofinal in ν.

In case (A), first let $q = O_P$, if $\alpha = \inf S^0$; if α immediately succeeds β in S^0, let $q = p(\beta)$ (so range $s^q = \gamma(\beta)\cdot\theta(\beta)$), while if α is a limit in S^0, let q be an upper bound for $\{p(\beta):\beta\in S^0\in\mu\}$ with range $s^q = \bigcup_{\beta\in S^0\cap\alpha}$ range $s^{p(\beta)} = \bigcup_{\beta\in S^0\cap\alpha} p(\beta) = \gamma(\alpha)$. (This is always possible by strong κ-directed closure.) In case (A1) it suffices to extend q to obtain p^* with $\gamma(\alpha)\cdot\theta(\alpha)\subseteq$ range s^{p^*}. This is always possible by (3.14). Then, if $p^* = \bar{p}^*(s^{p^*})$, we set $\bar{p}^*(\alpha) = \bar{p}^*$, $\rho(\alpha) = \lg\bar{p}^*$.

In case (A2) we use Indiscernibility and Amalgamation to guarantee that $p^*(\alpha)\geqq p(\bar{\nu},\nu)$. Let $\bar{\alpha} = \alpha_{\bar{\nu}}$. Note that $q\geqq p(\bar{\alpha})$. Let $p = p(\bar{\alpha})$, $p' = q$ and apply (6.3.1) to p, p', where h is defined as follows: $h \restriction \gamma(\bar{\alpha})\cdot\theta(\alpha) = f_{\bar{\nu}\nu}$, and for $\xi < \lg q - \gamma(\bar{\alpha})\cdot\theta(\bar{\alpha})$ (ordinal subtraction), let $h((\gamma(\bar{\alpha})\cdot\theta(\bar{\alpha}))+\xi) = (\gamma(\alpha)\cdot\nu)+\xi$. Then $h \restriction \gamma(\bar{\alpha})\cdot\bar{\alpha} = \mathrm{id}\restriction\gamma(\bar{\alpha})\cdot\bar{\alpha}$, and $h(\gamma(\bar{\alpha})\cdot\bar{\alpha}) = \gamma(\alpha)\cdot\alpha$. Thus, s^q, range h have the strong Δ-property, so, letting $\bar{q}\in\mathcal{T}$ be such that $q = \bar{q}(s^q)$, we have, by Amalgamation, that q, $\bar{q}(h)$ are compatible. Let q' be a common extension. Then, as in (A1), extend q' to obtain p^* with $\gamma(\alpha)\cdot\theta(\alpha)\subseteq s^{p^*}$, etc.

In cases (B)–(E), we first note that since $S'\neq\varnothing$, α is a limit in S^0, and so $\gamma(\alpha) = \bigcup_{\beta\in S^0\cap\alpha}\gamma(\beta)$. We let $\theta' = (\max S'_\alpha)+\alpha$, if S'_α has a largest element, and $\theta' = \sup S'_\alpha$ if not. Thus, in case (B), $\theta' = \theta(\alpha)$, in case (D), $\theta' = \tau+\alpha$, in case (E), $\theta' = \nu$.

In cases (B)–(E1), we start by taking q to be an upper bound for $D = \{p(\bar{\eta},\eta):\bar{\eta}\dashv\eta,\ \eta\in S'_\alpha\}$, with $s^q = \mathrm{id}\restriction\gamma(\alpha)\cdot\theta'$; this is always possible by strong μ-directed closure, since D is directed by (6.4.1)(c), and since $\bigcup_{p\in D}$ range $s^p = \gamma(\alpha)\cdot\theta'$ (this last is verified easily, because $\gamma(\alpha) = \bigcup_{\beta\in S^0\cap\alpha}\gamma(\beta)$).

Thus, if $q'\geqq q$, then (2′) holds when we replace $p^*(\alpha)$ by q', and (3′) holds for those $\eta\in S'_\alpha$, when we replace $p^*(\alpha)$ by q'. But this means that in case (B) we can simply take $p^*(\alpha) = q$. In case (C) we proceed as in case (A1).

In cases (D), (E1), we use (6.3.1) and Amalgamation. In case (D), recall that $\theta' = \tau+\alpha$. Let $\bar{\alpha} = \alpha_{\bar{\nu}}$. By (M1)(d), $\bar{\nu}$ is an immediate successor in $S_{\bar{\alpha}}$, say of $\bar{\tau}$. Again, by (M1)(d), $\pi_{\bar{\nu}\nu}(\bar{\tau}) = \tau$, and so by (M2), $\bar{\tau}\dashv\tau$, $\pi_{\bar{\tau}\tau}\restriction\bar{\tau}$ (whence, $f_{\bar{\tau}\tau}\restriction\gamma(\bar{\alpha})\cdot\bar{\tau} = f_{\bar{\nu}\nu}\restriction\gamma(\bar{\alpha})\cdot\bar{\tau}$). Also, $p(\bar{\tau},\tau) = \bar{p}(\bar{\alpha})(f_{\bar{\tau},\tau})\leqq q = \bar{q}(s^q)$, and $p(\bar{\nu},\nu) = \bar{p}(\bar{\alpha})(f_{\bar{\nu},\nu})$. We shall define $h:\gamma(\alpha)\cdot\theta'\to\gamma(\alpha)\cdot(\nu+\alpha)$ such that $h\restriction\gamma(\alpha)\cdot(\tau+\bar{\alpha}) = \mathrm{id}\restriction\gamma(\alpha)\cdot(\tau+\bar{\alpha})$, $h(\gamma(\alpha)\cdot(\tau+\bar{\alpha})) = \gamma(\alpha)\cdot(\tau+\alpha)$, $h\circ f_{\bar{\tau}\tau} = f_{\bar{\nu}\nu}$. To define h, let $\sigma = \gamma(\alpha)\cdot\tau$, $\psi = \bar{\nu}-\bar{\tau}$, $\sigma' = \gamma(\alpha)\cdot(\tau+\psi)$, $\psi' = \theta'-\sigma'$. Then, set $h\restriction\sigma = \mathrm{id}\restriction\sigma$; for $\bar{\xi}<\psi$, $\zeta<\gamma(\alpha)$, let $h((\gamma(\alpha)\cdot(\tau+\bar{\xi}))+\zeta) = (\gamma(\alpha)\cdot(\tau+\pi_{\bar{\nu}\nu}(\bar{\xi})))+\zeta = (\gamma(\alpha)\cdot\pi_{\bar{\nu}\nu}(\bar{\tau}+\bar{\xi}))+\zeta$; finally, for $\xi<\psi'$, set $h(\sigma'+\xi) = (\gamma(\alpha)\cdot\nu)+\xi$. So, by (6.3.1), $\bar{q}(h)\geqq p(\bar{\nu},\nu)$. By the Amalgamation

Property q, $\bar{q}(h)$ are compatible, so let q' be a common extension. Then work as in (A1).

In case (E1), recall that $\theta' = \nu$. Let $\bar{\alpha} = \alpha_{\bar{\nu}}$, and let $\lambda = \sup \operatorname{range} \pi_{\bar{\nu}\nu} < \nu$. So, by (M6), $\lambda \in S_\alpha$, $\bar{\nu} \dashv\!\!3\ \lambda$, $\pi_{\bar{\nu}\lambda} = \pi_{\bar{\nu}\nu}$ (and so $f_{\bar{\nu}\lambda} \mid \gamma(\alpha)\cdot\bar{\nu} = f_{\bar{\nu}\nu} \mid \gamma(\bar{\alpha})\cdot\bar{\nu}$). Further, $q = \bar{q}(s^q) \geqq p(\bar{\nu},\lambda) = \bar{p}(\bar{\alpha})(f_{\bar{\nu}\lambda})$. We set $p = p(\bar{\nu},\lambda)$, $p' = q$ and we apply (6.3.1) to h which we define as follows: $h \mid \gamma(\alpha)\cdot\lambda = \mathrm{id} \mid \gamma(\alpha)\cdot\lambda$; for $\xi < \gamma(\alpha)\cdot(\nu-\lambda)$, $h((\gamma(\alpha)\cdot\lambda)+\xi) = (\gamma(\alpha)\cdot\nu)+\xi$. Thus $h \circ f_{\bar{\nu}\lambda} = f_{\bar{\nu}\nu}$, and so by (6.3.1) $\bar{q}(h) \geqq p(\bar{\nu},\nu)$. Also, by construction θ', range h have the strong Δ-property so by Amalgamation q, $\bar{q}(h)$ are compatible. Let $q' \geqq q, \bar{q}(h)$; work as in (A1).

Case (E2) is by far the most subtle. Recall that $\theta' = \nu$. We use (6.3.2), and we let $(\bar{\eta}_i : i < \lambda)$, $(\tau_i : i < \lambda)$, $(\eta_i : i < \lambda)$, $(\eta_i^* : i < \lambda)$, $(\alpha_i : i < \lambda)$, $(\eta_{i,j}^* : i < j < \lambda)$ be as in (6.3.2).

We shall construct an increasing chain of conditions $(q_i : i < \lambda)$ where $q_i = \bar{p}(\alpha_i)(h_i)$. Before defining h_i, note that $\theta(\alpha) = \nu + \alpha$, $\theta(\alpha_i) = \eta_i^* + \alpha_i$ (since ν, η_i^* are immediate successors in $\dashv\!\!3$, and so $\nu = \max S_\alpha$, $\eta_i^* = \max S_{\alpha_i}$). We shall have $h_i : \rho(\alpha_i) \to \gamma(\alpha)\cdot\theta(\alpha)$. In order to define h_i, let $\sigma = \gamma(\alpha_i)\cdot\eta_i^*$, $\psi = \bar{\nu} - \bar{\eta}_i$, $\sigma' = \gamma(\alpha_i)\cdot(\eta_i^* + \psi)$, $\psi' = \rho(\alpha_i) - \sigma'$. Then set $h_i \mid \sigma = f_{\eta_i^*\eta_i} \mid \sigma$; for $\bar{\xi} < \psi$, $\zeta < \gamma(\alpha_i)$, set $h_i((\gamma(\alpha_i)\cdot(\eta_i^* + \bar{\xi})) + \zeta) = (\gamma(\alpha)\cdot\pi_{\bar{\nu}\nu}(\bar{\eta}_i + \bar{\xi})) + \zeta$; finally, for $\xi < \psi'$, set $h_i(\sigma' + \xi) = (\gamma(\alpha)\cdot\nu) + \xi$.

We must show:

(a) for $i < j < \lambda$, $p(\bar{\nu},\nu) \leqq q_i \leqq q_j$,

(b) if $\tau \in S'_\alpha$, $\bar{\tau} \dashv\!\!3\ \tau$, then for some $i < \lambda$, $p(\bar{\tau},\tau) \leqq q_i$.

Assuming we've proved (a), (b), we note that $\bigcup_{i<\lambda} \operatorname{range} h_i = \gamma(\alpha)\cdot(\nu+\alpha)$. We take $p^*(\alpha)$ to be an upper bound for the q_i's, and so clearly (1′), (2′), (3′) of (6.4.2) hold. So, we turn to proving (a), (b). Our main tool is (6.3.1).

For (a), let $i < j < \lambda$. To see that $p(\bar{\nu},\nu) \leqq q_i$, note that $p(\bar{\eta}_i, \eta_i^*) = \bar{p}(\bar{\alpha})(f_{\bar{\eta}_i\eta_i^*}) \leqq \bar{p}(\alpha_i)\,(\mathrm{id} \mid \rho_i) = p(\alpha_i)$. However, by construction, $h_i \circ f_{\bar{\eta}_i\eta_i^*} = f_{\bar{\nu}\nu}$, so, by (6.3.1), $p(\bar{\nu},\nu) \leqq \bar{p}(\alpha_i)(h_i) = q_i$. To see that $q_i \leqq q_j$, note that $p(\eta_i^*, \eta_{i,j}^*) \leqq \bar{p}(\alpha_j)(\mathrm{id} \mid \rho_j) = p(\alpha_j)$. By construction, $h_j \circ f_{\eta_i^*\eta_{i,j}^*} = h_i$, so by (6.3.1), $q_i = \bar{p}(\alpha_i)(h_i) \leqq \bar{p}(\alpha_j)(h_j) = q_j$.

For (b), using (M5) and Lemma (6.3.2)(b), given $\bar{\tau} \dashv\!\!3\ \tau$, $\tau \in S'_\alpha$, we can find a successor ordinal i such that $\alpha_i > \alpha_{\bar{\tau}}$, $\eta_i = \tau_i > \tau$, $\tau \in \operatorname{range} \pi_{\eta_i^*\eta_i}$. Let τ' be such that $\pi_{\eta_i^*\eta_i}(\tau') = \tau$. So, by (M2), $\bar{\tau} \dashv\!\!3\ \tau' \dashv\!\!3\ \tau$, $\pi_{\tau'\tau} = \pi_{\eta_i^*\eta_i} \mid \tau'$. Also $p(\alpha_i) = \bar{p}(\alpha_i)(\mathrm{id} \mid \rho_i) \geqq \bar{p}(\bar{\alpha})(f_{\bar{\tau}\tau'}) = p(\bar{\tau},\tau')$. Also, range $f_{\bar{\tau}\tau'} \subseteq \gamma(\alpha_i)\cdot(\tau' + \alpha_i)$, and $\pi_{\eta_i^*\eta_i} \mid \alpha_i = \mathrm{id} \mid \alpha_i$. Thus

$$f_{\tau'\tau} \mid \gamma(\alpha_i)\cdot(\tau'+\alpha_i) = f_{\eta_i^*\eta_i} \mid \gamma(\alpha_i)\cdot(\tau'+\alpha_i) = h_i \mid \gamma(\alpha_i)\cdot(\tau'+\alpha_i).$$

This means that $f_{\bar{\tau}\tau} = f_{\tau'} \circ f_{\bar{\tau}\tau'} = h_i \circ f_{\bar{\tau}\tau'}$. Thus, applying (6.3.1) to $p = p(\bar{\tau}, \tau')$, $p' = p(\alpha_i)$ and h_i, we have $p(\bar{\tau}, \tau) = \bar{p}(\bar{\alpha})(f_{\bar{\tau}\tau}) \leqq \bar{p}(\alpha_i)(h_i) = q_i$. This completes the proof of (b), the treatment of case (E2) and the proof of (6.1).

References

1. J. P. Burgess, *Infinitary Languages and Descriptive Set Theory*, Ph.D. Thesis, University of California, Berkeley, 1974.
2. J. P. Burgess, *Consistency proofs in model theory, a contribution to Jensenlehre*, Ann. Math. Logic **14** (1978), 1–12.
3. K. Devlin, *Aspects of Constructibility*, Lecture Notes in Math. **354**, Springer-Verlag, Berlin, 1973.
4. K. Devlin, to appear.
5. J. Gregory, *Higher Souslin trees*, J. Symb. Logic **41** (1976), 663–671.
6. T. Jech, *A game theoretic property of Boolean algebras*, in *Logic Colloquium '77* (A. MacIntyre, L. Pacholski and J. Paris, eds.), North Holland, 1978, pp. 135–144.
7. R. B. Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic **4** (1972), 229–308.
8. R. Laver, *Making the supercompactness of κ indestructible under κ-directed closed forcing*, Isr. J. Math. **29** (1978), 385–388.
9. R. Laver and S. Shelah, *The $\aleph_2$-Souslin-Hypothesis*. Trans. Am. Math. Soc. **264** (1981), 411–418.
10. S. Shelah, *Models with second-order properties* IV: *A general method and eliminating diamonds*, to appear.
11. S. Shelah and L. Stanley, *Generalized Martin's Axiom and Souslin's Hypothesis for higher cardinals*, Isr. J. Math. **43** (1982), 225–236.
12. L. Stanley, *A Short Course on Gap-*1 *Morasses with a Review of the Fine Structure of L*, Proceedings of the Cambridge Summer School in Set Theory, to appear.
13. L. Stanley, *Large cardinals and morasses*, to appear.
14. D. Velleman, Ph.D. Thesis, University of Wisconsin, Madison, 1980.

Institute of Mathematics
The Hebrew University of Jerusalem
Jerusalem, Israel

Maths Pures-Les Cezeaux
Université de Clermont II
63170 Aubiere, France

Current address of second author
Department of Mathematics
Lehigh University
Bethlehem, PA 18015 USA